Plan: - Linearize systems which are almost linear at an isolated critical point.
- Use the linearized system to predict the behavior of the linear system near an isolated critical pt.

Taylor: $f(x,y)$ nice, (x_0,y_0) given

$$f(x_0+u,\ y_0+v) = \overbrace{f(x_0,y_0)}^{\text{const.}} + \overbrace{\partial_x f\Big|_{(x_0,y_0)} u + \partial_y f\Big|_{(x_0,y_0)} v}^{\text{linear terms}} + \underbrace{r(u,v)}_{\text{error}}$$

u,v small

error small relative to $|(u,v)|$

where $$\lim_{(u,v)\to(0,0)} \frac{r(u,v)}{\sqrt{u^2+v^2}} = 0$$

Ex: $f(x,y) = e^{-x^2-y^2}$ at $(x_0,y_0) = (1,0)$

$$\partial_x f = -2x\,e^{-x^2-y^2}, \quad \partial_y f = -2y\,e^{-x^2-y^2}$$

$$f(1+u,v) = e^{-1} + (-2e^{-1})u + 0\cdot v + \underbrace{r(u,v)}_{\text{error}}.$$

//

We're looking at systems

$$① \begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \quad \text{(autonomous)}$$

Let (x_0,y_0) be a CP. Apply Taylor's f. to ①, at (x_0,y_0)

$$\begin{cases} \dfrac{du}{dt} = \partial_x f\big|_{(x_0,y_0)}\, u + \partial_y f\big|_{(x_0,y_0)}\, v + r(u,v) \\[2ex] \dfrac{dv}{dt} = \partial_x g\big|_{(x_0,y_0)}\, u + \partial_y g\big|_{(x_0,y_0)}\, v + s(u,v) \end{cases}$$

$f(x_0,y_0)$, $g(x_0,y_0) = 0$ at CP.

For u, v small enough, $r(u,v)$, $s(u,v)$ negligible; truncate:

linear system.
$$\begin{cases} \dfrac{du}{dt} = \partial_x f\big|_{(x_0,y_0)}\, u + \partial_y f\big|_{(x_0,y_0)}\, v \\[2ex] \dfrac{dv}{dt} = \partial_x g\big|_{(x_0,y_0)}\, u + \partial_y g\big|_{(x_0,y_0)}\, v \end{cases}$$

Linearized system assoc. to (1) at the CP (x_0,y_0).

Matrix of the system: Jacobian

$$J(x_0,y_0) = \begin{bmatrix} \partial_x f\big|_{(x_0,y_0)} & \partial_y f\big|_{(x_0,y_0)} \\ \partial_x g\big|_{(x_0,y_0)} & \partial_y g\big|_{(x_0,y_0)} \end{bmatrix}$$

Can write $\underline{\underline{u}}' = \underline{\underline{J}}\, \underline{\underline{u}}$. $\quad \underline{\underline{u}} = \begin{bmatrix} u \\ v \end{bmatrix}$

Def'n

If (x_0, y_0) is an isolated CP for (1), and $(0,0)$ is an isolated CP for the linearized system [0 not an e-value of linearized system] and $r(u,v), s(u,v)$ negligible $\left(\lim \frac{r(u,v)}{\sqrt{u^2+v^2}} = 0 \right)$

then (1) is an Almost Linear System (ALS) at (x_0, y_0).

"well approximated by linear system near (x_0,y_0)"

Non example:

$$\begin{cases} \frac{dx}{dt} = x - y^2 \\ \frac{dy}{dt} = x + y^2 \end{cases}$$

isolated CP $(0,0)$ but $(0,0)$ not isolated for linearized system.

$$\frac{du}{dt} = u \qquad \frac{dv}{dt} = u$$

Ex:

$$\frac{dx}{dt} = e^{x+y} - 1$$

$$\frac{dy}{dt} = x^3 + y.$$

CP: $\underbrace{e^{x+y} - 1 = 0}$ $\qquad x^3 + y = 0$

$$x+y=0 \Rightarrow x=-y$$

So: $x^3 - x = 0 \Rightarrow x = 1, 0, -1.$

$$\Rightarrow \underbrace{(0,0),\ (1,-1),\ (-1,1)}_{\text{isolated (finitely many)}}$$

Linearize at each.

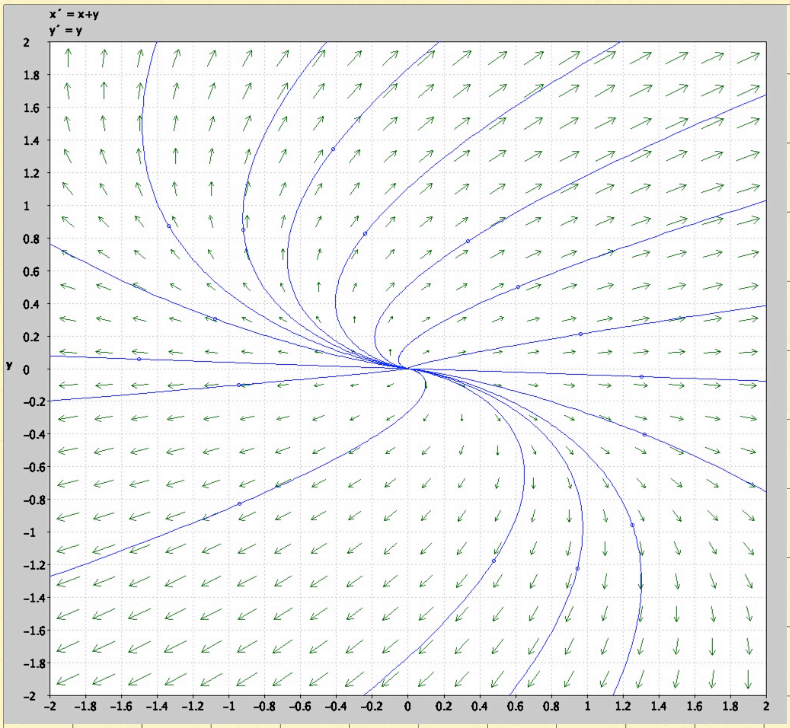
$$J(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ 3x^2 & 1 \end{bmatrix}$$

At $(0,0)$: $\underline{\underline{u}}' = \underline{\underline{J}}(0,0)\,\underline{\underline{u}}$

$$\Rightarrow \begin{cases} u' = u+v \\ v' = v \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

E-values: $(1-\lambda)^2 = 0 \Rightarrow \lambda = 1$ e-value, repeated.

Phase plane p: Nodal source



At $(1,-1)$: $\underline{\underline{u}}' = J(1,-1)\,\underline{\underline{u}}$

$\Rightarrow \begin{cases} u' = u + v \\ v' = 3u + v \end{cases}$ linearized system at $(1,-1)$

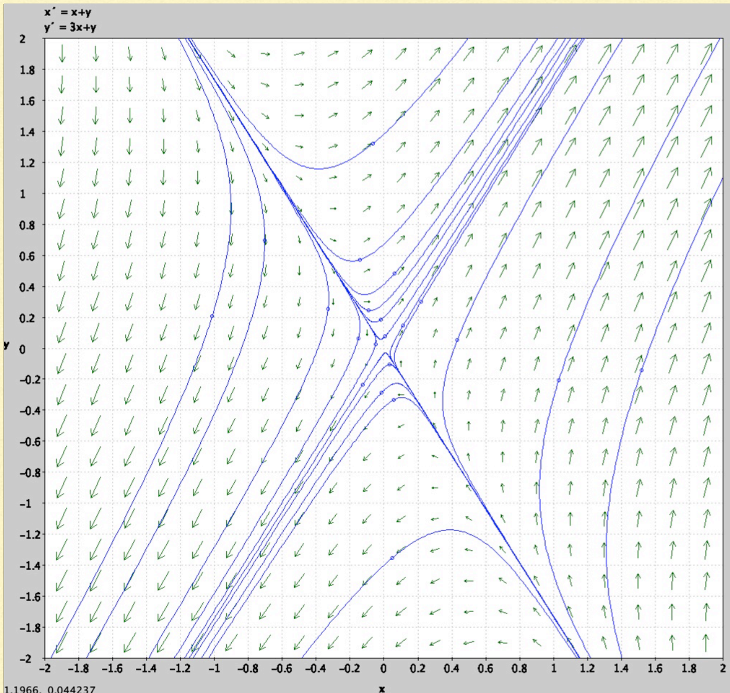
$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$

E-values:

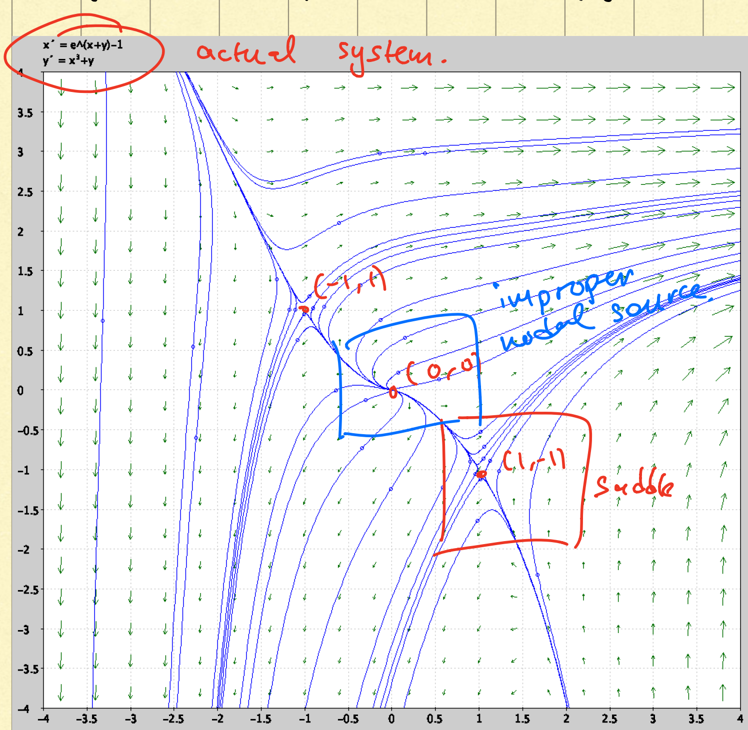
$(\lambda - 1)^2 - 3 = 0$

$\Rightarrow \lambda - 1 = \pm\sqrt{3} \Rightarrow \lambda = 1 + \sqrt{3}$, $\lambda = 1 - \sqrt{3}$ opposite signs.

Phase plane p: saddle.



At $(-1,1)$: exercise.

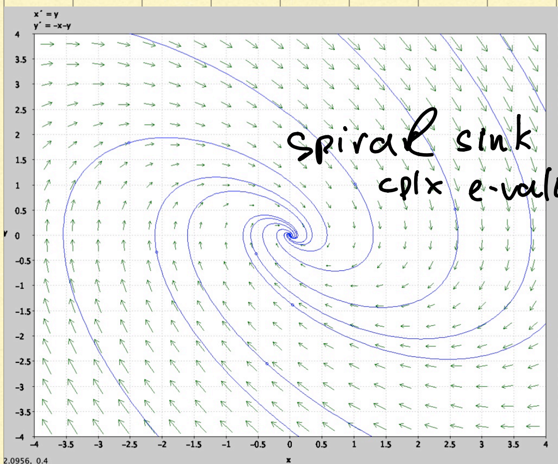


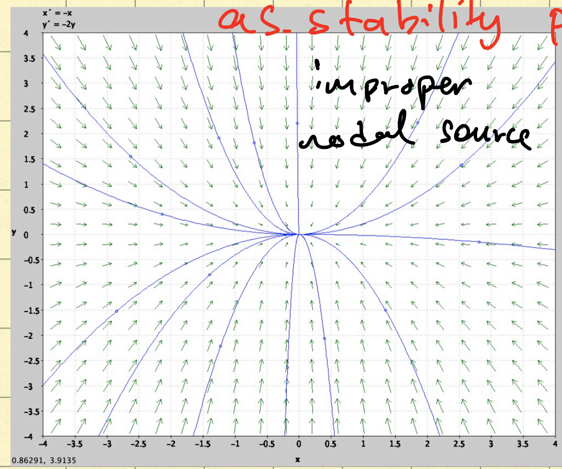
So: linearized system helps understand non-linear system near C.P.

Can think of non-linear ALS near a CP as a perturbation of the linearized system. Recall stability properties of linear systems:

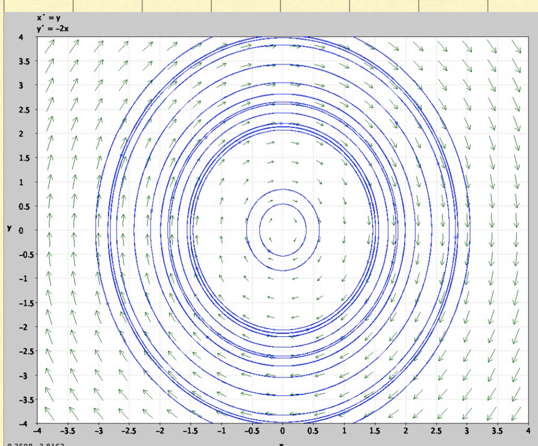
Asymptotically stable if $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$

as. stability preserved





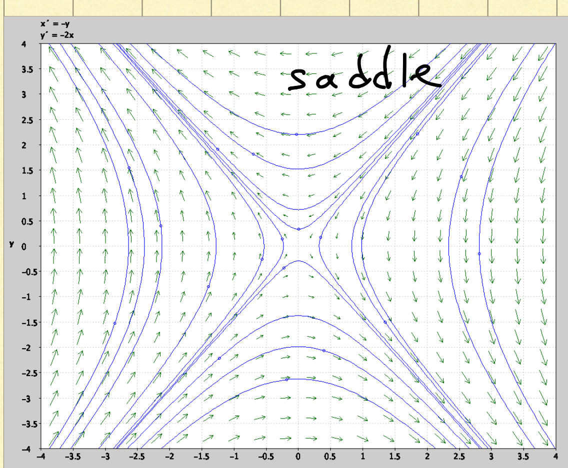
Stable but not asymptotically stable if $\underbrace{\lambda_{1,2} = \pm q i}_{Re(\lambda_{1,2}) = 0}$

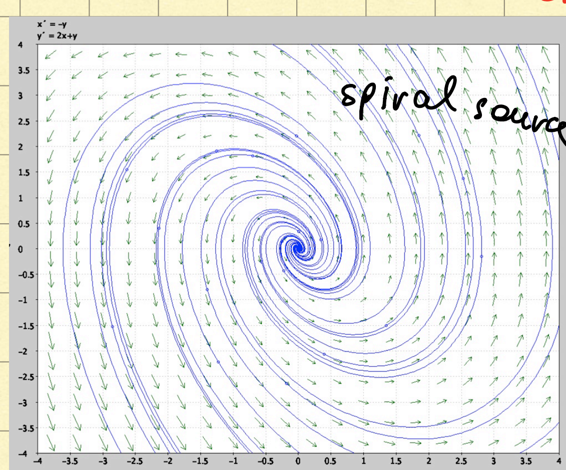


Under perturbation we might not have $Re(\lambda_{1,2}) = 0$ anymore $\longrightarrow$ spiral sink or source

Unstable if $Re(\lambda_1) > 0$ or $Re(\lambda_2) > 0$

instability preserved





Stability is encoded in real pt of e-values.

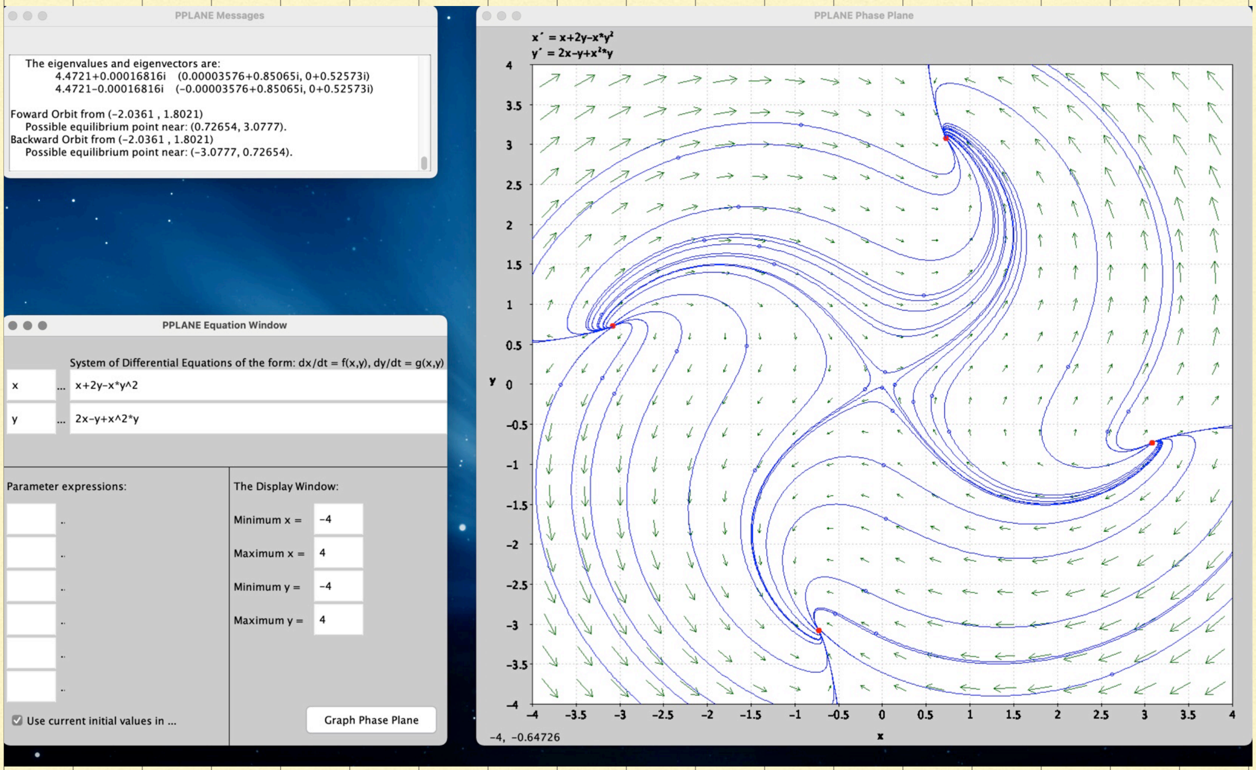
Q: Are properties like $Re(\lambda_{1,2}) > 0$ preserved under perturbations?

Principle: - inequality ($Re(\lambda) > 0$, or $Re(\lambda) < 0$) preserved

- equality is not ($Re(\lambda) = 0$, $\lambda_1 = \lambda_2$)

can break under perturbation.

Next time: Stability properties from linear system → perturbed linear system behave similar way to linearized → nonlinear.


PPLANE Messages
The eigenvalues and eigenvectors are:
4.4721+0.00016816i (0.00003576+0.85065i, 0+0.52573i)
4.4721-0.00016816i (-0.00003576+0.85065i, 0+0.52573i)
Foward Orbit from (-2.0361 , 1.8021)
Possible equilibrium point near: (0.72654, 3.0777).
Backward Orbit from (-2.0361 , 1.8021)
Possible equilibrium point near: (-3.0777, 0.72654).
PPLANE Equation Window
System of Differential Equations of the form: dx/dt = f(x,y), dy/dt = g(x,y)
x ... x+2y-x*y^2
y ... 2x-y+x^2*y
Parameter expressions:
The Display Window:
Minimum x = -4
Maximum x = 4
Minimum y = -4
Maximum y = 4
Use current initial values in ...
Graph Phase Plane
PPLANE Phase Plane
x' = x+2y-x*y²
y' = 2x-y+x²*y
y
x
-4, -0.64726