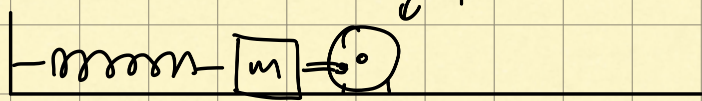
Plan: Laplace transform & inverse Laplace transform

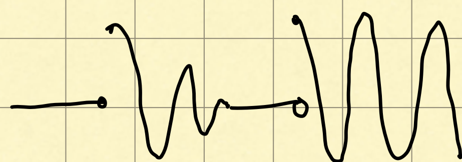
produces external force



$$m x'' = \underbrace{-cx'}_{\text{damping}} \; \overbrace{-kx}^{\text{Hooke's}} \; \overbrace{+f(t)}^{\text{external force.}}$$

Saw in 266/262 using characteristic eqn / undetermined coef.

New tool: can deal w/ more complicated functions, e.g. impulses, piecewise cont. fcts.



Laplace transform: given $f(t)$ defined for $t \geq 0$

Its Laplace transform is given by

$$F(s) = \underbrace{\mathcal{L}\{f(t)\}}_{\text{depends on } s} = \int_0^\infty e^{-st} f(t)\, dt = \lim_{M \to \infty} \int_0^M e^{-st} f(t)\, dt$$

upper case

for the $s \in \mathbb{R}$ for which the integral converges.

functions of t $\xrightarrow{\text{Laplace}}$ functions of s

Ex 1. $f(t) = e^{at}$, $t \geq 0$, a real

$\mathcal{L}\{e^{at}\} = \,?$, for what s does it make sense?

Ex 2: $f(t) = 1$, same.

Ex 1: $\lim_{M\to\infty} \int_0^M e^{-st} e^{at}\,dt = \lim_{M\to\infty} \int_0^M e^{(a-s)t}\,dt$

$$\overset{a\neq s}{=} \lim_{M\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^M$$

$$= \lim_{M\to\infty} \left(\frac{e^{(a-s)M}}{a-s} - \frac{1}{a-s} \right)$$

converges if $s>a$ to $\frac{1}{s-a}$

goes to ∞ otherwise.

So: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$. //

Ex 2: take $a=0$ in Ex 1:

$\mathcal{L}\{1\} = \frac{1}{s}$, $s>0$. //

Rmk: Laplace makes sense for functions which satisfy

$$|f(t)| \leq M e^{ct}$$

for large t, and for some M, c.

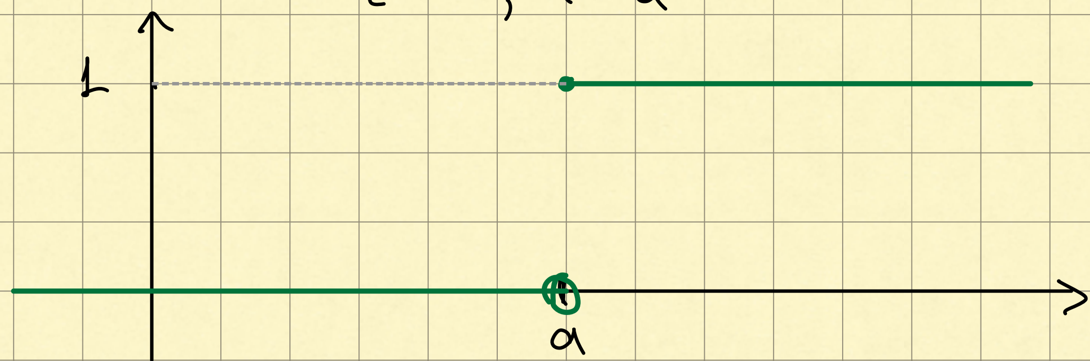
Then $\mathcal{L}\{f\}$ is defined for $s > c$

Ex: e^t, t^{25}, $\sinh(t) + \sin(t) + e^{28t}$

Non-ex: e^{t^2} (bad)

Ex 3: Step functions.

$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \qquad a \in \mathbb{R}$$



If $a=0$ can write $u(t) \doteq u_0(t)$

Take $a>0$:

$$\mathcal{L}\{u_a(t)\} = \lim_{M\to\infty} \int_0^M e^{-st}\, u_a(t)\, dt$$

$$= \lim_{M\to\infty} \left(\int_0^a e^{-st}\, 0 \, dt + \int_a^M e^{-st}\, 1 \, dt \right) \qquad 0$$

$$= \lim_{M\to\infty} \left[\frac{e^{-st}}{-s} \right]_a^M = \frac{e^{-as}}{s}$$

for $s>0$ //

Read: Gamma fcts p. 439.

Laplace tr. is linear!

$$\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$$

constants.

Ex: $\mathcal{L}\{3t^4 + 5\cosh(3t)\}$

$= 3\mathcal{L}\{t^4\} + 5\mathcal{L}\{\cosh(3t)\}$

$= 3\dfrac{4!}{s^5} + 5\dfrac{s}{s^2-9}$

table

//

If $F(s) = \mathcal{L}\{f(t)\}$ then $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is the Inverse Laplace Transform.

Process: given $F(s)$, break it down to simpler functions for which we can find $\mathcal{L}^{-1}$ using tables.

Ex:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}+\frac{1}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= t + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\}$$

$$= t + \frac{1}{2}\sinh(2t) \quad //$$