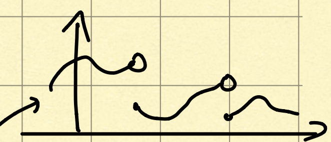
Plan: - More examples on inverse Laplace
- Solving 1st & 2nd order eq's using the Laplace transform.
- Practice partial fractions.

Recall: $f(t)$, $t \geq 0$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Said: if $f(t)$ piecewise continuous



and there are M, c such that for large t

$$|f(t)| \leq Me^{ct}$$

then $\mathcal{L}\{f(t)\}$ is defined for $s > c$ &

$$\mathcal{L}\{f(t)\}(s) \xrightarrow{s\to\infty} 0$$

For inverse Laplace: Break given $F(s)$ into a sum of functions for which we can use table to find inv. Laplace.

Ex: $F(s) = \dfrac{1}{s(s^2+4s+3)}$. Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$

Partial Fractions

$$s^2+4s+3 = (s+3)(s+1)$$

$$\frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+3)(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}$$

For A: Multiply by s, set $s=0$

$$\frac{1}{(s+3)(s+1)} = A + \frac{Bs}{s+3} + \frac{Cs}{s+1}$$

$\overset{s=0}{\Rightarrow}$ $A = \frac{1}{3}$

For B:

$$\frac{1}{s(s+1)} = \frac{1}{3}\frac{1}{s}(s+3) + B + \frac{C}{s+1}(s+3)$$

$\xrightarrow{s=-3}$ $B = \frac{1}{6}$

Similarly: $C = -\frac{1}{2}$ (check!)

$$F(s) = \frac{1}{3}\frac{1}{s} + \frac{1}{6}\frac{1}{s+3} - \frac{1}{2}\frac{1}{s+1}$$

$\overset{\text{table}}{\Rightarrow}$ $f(t) = \frac{1}{3}\cdot 1 + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}$ //

Property

→ f dif'ble $|f(t)| \leq Me^{ct}$ as before

$$\underbrace{\mathcal{L}\{f'(t)\}(s)}_{\text{fct of } s} = s\underbrace{\mathcal{L}\{f(t)\}(s)}_{\text{fct of } s} - \underbrace{f(0)}_{\text{const.}} \quad (*)$$

Dif'tion ⟶ multiplication by s.

Pf: $\int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t) \Big|_0^\infty - (-s) \int_0^\infty e^{-st} f(t)\,dt$

uses growth condition

$= -f(0) + s \int_0^\infty e^{-st} f(t)\,dt$ //

Use (*) to solve Initial Value Problems.

Ex: $\begin{cases} 4x' + 3x = 1 & (1) \\ x(0) = 0 \end{cases}$

Use (*) $\longrightarrow$ find $X(s) = \mathcal{L}\{x(t)\} \longrightarrow$ take inverse Laplace to find $x(t)$.

Take Laplace on (1)

$$4\,\mathcal{L}\{x'(t)\} + 3\,\mathcal{L}\{x(t)\} = \mathcal{L}\{1\}$$

$$4\big(sX(s) - \underbrace{x(0)}_{=0}\big) + 3X(s) = \frac{1}{s}$$

$$\Rightarrow X(s)(4s+3) = \frac{1}{s}$$

$$\Rightarrow X(s) = \frac{1}{s(4s+3)}$$

Use: $\mathcal{L}\left\{\frac{1}{s-a}\right\} = e^{at}$

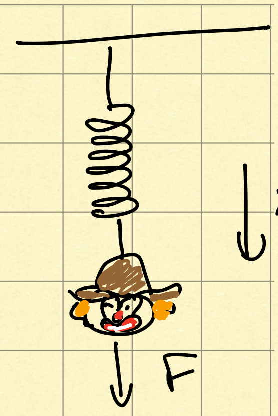
So: $x(t) =$

$$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{3}{4}}\right\}$$

$$\Rightarrow x(t) = \frac{1}{3} - \frac{1}{3}e^{-\frac{3}{4}t}$$ //.

Ex 2:

$$\begin{cases} x'' + 9x = 1 \\ x(0) = 0, \quad x'(0) = 1 \end{cases}$$



Solve w/ Laplace tr!

Info: $$\frac{s^2 + cs + d}{(s-a)(s^2+b^2)} = \frac{A}{s-a} + \frac{Bs + C}{s^2+b^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \quad \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

$$\mathcal{L}\{x'' + 9x\} = \mathcal{L}\{1\} \Rightarrow$$

$$\Rightarrow s\mathcal{L}\{x'\} - x'(0) + 9\mathcal{L}\{x\} = \frac{1}{s}$$

$$\Rightarrow s^2\mathcal{L}\{x\} - s\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=1} + 9\mathcal{L}\{x\} = \frac{1}{s}$$

$$\Rightarrow X(s)(s^2+9) = 1 + \frac{1}{s}$$

$$\Rightarrow X(s) = \frac{s+1}{s(s^2+9)}$$

$$\frac{s+1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} = \frac{As^2 + 9A + Bs^2 + Cs}{s(s^2+9)}$$

$$\Rightarrow \underset{B = -\frac{1}{9}}{A + B = 0}, \quad \underset{A = \frac{1}{9}}{9A = 1}, \quad C = 1$$

$$\Rightarrow X(s) = \frac{1}{9}\cdot\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9} + \frac{1}{s^2+9}$$

$$\overset{\text{table}}{\Longrightarrow} \quad x(t) = \frac{1}{9} - \frac{1}{9}\cos(3t) + \frac{1}{3}\sin(3t) \quad //.$$