Plan: finish 7.2
Partial fractions

Laplace Transform of Integrals

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

or:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}(\tau)\,d\tau = \int_0^t f(\tau)\,d\tau$$

Integration $\xleftrightarrow{\text{Laplace}}$ multiplication by $\frac{1}{s}$.

Ex: $X(s) = \dfrac{1}{s(s^2+9)}$. Find $x(t) = \mathcal{L}^{-1}\{X(s)\}$

1st way: partial fractions.

2nd: take $F(s) = \dfrac{1}{s^2+9}$

dummy variable

$$\Rightarrow \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}(\tau)\,d\tau$$

$$\overset{\text{table}}{=} \int_0^t \frac{1}{3}\sin(3\tau)\,d\tau = \left.\frac{-\cos(3\tau)}{9}\right|_0^t$$

$$= \frac{-\cos(3t)}{9} + \frac{1}{9} \quad //$$

Partial fractions

Decompose Rational Functions:

$$R(s) = \frac{P(s)}{Q(s)}$$, where $P(s)$, Q_s polynomials, w/ $\deg P < \deg Q$

Ex: $\frac{s^2 - 3s + 1}{s^4 - 2s^3 + 4}$, $\frac{s-1}{s^2+2}$

deg 2, deg 4, deg 1, deg 2

If $\deg P > \deg Q$? ex: $\frac{s^3 - 3s^2 + 4s - 2}{s^2 + 1}$

W/ long division, if $\deg Q > \deg P$

$$P(s) = Q(s)\,\underbrace{\tilde{Q}(s)} + \underbrace{R(s)}$$

polynomials, $\deg R < \deg Q$

$$\frac{P}{Q} = \frac{Q\tilde{Q} + R}{Q} = \tilde{Q} + \frac{R}{Q}$$

polyn. $\deg R < \deg Q$.

Now assume $\deg P < \deg Q$:

Factor $Q(s)$ into linear factors $(s-a)^n$ — 0 for $s=a$

& irreducible quadratic factors $((s-a)^2 + b^2)^m$ $(b \neq 0)$ — $\neq 0$ for all real s

Ex: $Q(s) = s^3 + 9s = s(s^2+9)$

$$\underset{(s-0)^1}{s} \quad \underset{((s-0)^2+3^2)^1}{(s^2+9)}$$

$\to$ Part of decomposition corr. to $(s-a)^n$

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}$$

also take lower power of $\frac{1}{s-a}$

$\to$ Part of decomposition corr. to $((s-a)^2+b^2)^m$

$$\frac{A_1 s + B_1}{((s-a)^2+b^2)} + \dots + \frac{A_m s + B_m}{((s-a)^2+b^2)^m}$$

Ex 1. $\dfrac{s-1}{(s+1)(s^2-s-2)}$ partial fractions?

$$s^2 - s - 2 = 0 \Rightarrow s = \frac{1 \pm \sqrt{1+8}}{2}$$

$$= \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

2 real roots $\Rightarrow (s^2 - s - 2) = (s-2)(s+1)$

$$\frac{s-1}{(s+1)^2(s-2)} = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{s-2}$$

Multiply through by $(s+1)^2(s-2)$

$$s-1 = A_1(s-2)(s+1) + A_2(s-2) + A_3(s+1)^2 \qquad (1)$$

Find A_2: set $s=-1$

$$-2 = -3A_2 \Rightarrow A_2 = \frac{2}{3}$$

A_3: set $s=2$

$$1 = 9A_3 \Rightarrow A_3 = \frac{1}{9}.$$

For A_1: 1st way: know A_2, A_3, try to set anything other than $2, -1$. Take $s=0$

$$-1 = -2A_1 - 2A_2 + A_3$$

$$\Rightarrow -1 = -2A_1 - \frac{4}{3} + \frac{1}{9}$$

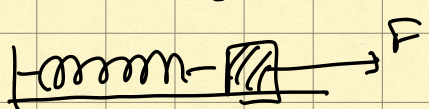
$$\Rightarrow A_1 = \dots = -\frac{1}{9}$$

2nd way: differentiate (1), set $s=-1$

$$1 = A_1(s-2) + A_1(s+1) + A_2 + 2(s+1)A_3$$

$$\overset{s=-1}{\Longrightarrow} 1 = -3A_1 + A_2 \Rightarrow A_1 = -\frac{1}{9} \qquad //$$

Ex: Spring-mass system w/ periodic external force



(2) $$x'' + 9x = \underbrace{5\cos(2t)}_{\text{external force}}$$

w/o external force:

$$x'' + 9x = 0 \Rightarrow \text{general sol'n}$$

$x(t) = A\cos(3t) + B\sin(3t)$

angular freq. $3 = \sqrt{9}$

Let $x(0) = x'(0) = 0$. Solve 2

$$s^2 X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{0} + 9X(s) = 5\frac{s}{(s^2+4)} \quad \text{table}$$

$$\Rightarrow X(s) = 5\frac{s}{(s^2+4)(s^2+9)}$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

irred. quadratic

1st way: Multiply by $(s^2+4)(s^2+9)$

$$s = \underbrace{As^3 + Bs^2 + 9As + 9B}_{(As+B)(s^2+9)} + Cs^3 + Ds^2 + 4Cs + 4D$$

match terms
$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 9A + 4C = 1 \\ 9B + 4D = 0 \end{cases}$$
solve

2nd way: Multiply by $(s^2+4)(s^2+9)$

$$s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

set $s = 2i$

$$2i = (2iA + B)\underbrace{(-4+9)}_{5} + 0$$

$$\Rightarrow \quad 10Ai + 5B = 2i$$

$$\Rightarrow \begin{cases} 10A = 2 & \text{(imaginary pts match)} \\ B = 0 & \text{(real pts match).} \end{cases}$$

For C, D same, set $s = 3i$.

Finish

//