Last time: convolution

Seen: Diffferentiation & integration in $t$ $\overset{\mathcal{L}}{\longleftrightarrow}$ multiplication / division by $s$.

1. $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$
2. $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

Now: multiplication / division by $t$ $\overset{\mathcal{L}}{\longleftrightarrow}$ dif'tial / integration in $s$.

3. $\mathcal{L}\{(-t)f(t)\} = F'(s)$ $\quad (F(s) = \mathcal{L}\{f(t)\})$
4. If $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists & is finite* then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

$$\iff f(t) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$

* if $f$ cont. at $0$, enough to have $f(0)=0$ for the functions we will work with

Non-ex: $f(t) = 1$

$$\mathcal{L}\left\{\frac{1}{t}\right\} = \int_0^\infty e^{-st}\frac{1}{t}\,dt$$

doesn't converge, $\frac{1}{t}$ blows up at $0$.

Ex: $\mathcal{L}\{t^2\cos(2t)\}$

Option 1: def'n.

$$\int_0^\infty e^{-st}\, t^2 \cos(2t)\,dt.$$ IBP (at least 2)

Option 2.

$$\boxed{\mathcal{L}\{(-t)f(t)\} = F'(s)}$$

$$\mathcal{L}\{t^2\cos(2t)\} = \mathcal{L}\{(-t)(-t)\cos(2t)\}$$

$$= \frac{d}{ds}\mathcal{L}\{(-t)\cos(2t)\} =$$

$$= \frac{d^2}{ds^2}\mathcal{L}\{\cos(2t)\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$$

$$= \ldots = \frac{2s(s^2-12)}{(s^2-4)^3}$$ //

Ex 4: $g(t) = \dfrac{e^t - e^{-t}}{t}$

Set $f(t) = e^t - e^{-t}$

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma}$$

Check that $\lim_{t\to 0^+}\dfrac{f(t)}{t}$ finite:

$$\lim_{t\to 0^+} \frac{f(t)}{t} \overset{0/0}{=} \lim_{t\to 0^+} \frac{f'(t)}{t'} = \lim_{t\to 0^+} \frac{e^t+e^{-t}}{1}$$

L'Hôpital

$$= 2 \quad ✓$$

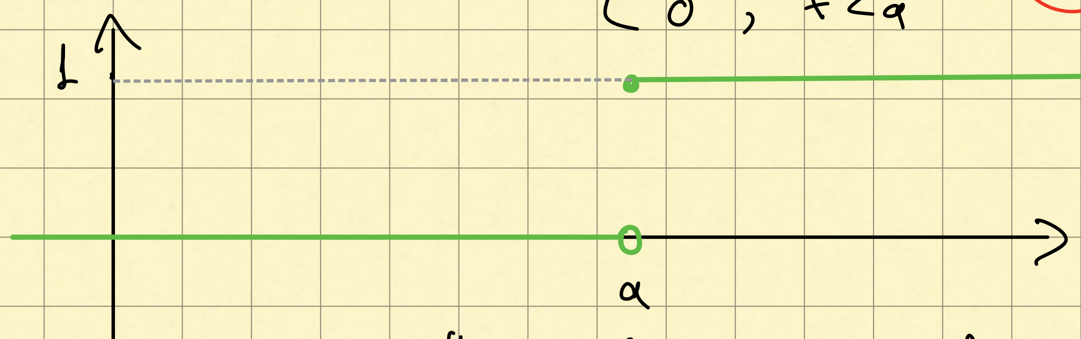
$$\mathcal{L}\left\{\frac{e^t-e^{-t}}{t}\right\} = \int_s^\infty \mathcal{L}\{e^t - e^{-t}\}(\sigma)\, d\sigma$$

$$= \int_s^\infty \frac{1}{\sigma-1} - \frac{1}{\sigma+1}\, d\sigma$$

$$= \ldots = \ln\frac{s+1}{s-1} \quad //$$

## $\boxed{7.5}$ Recall:

$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \quad (*)$$




If $a = 0$ write $u(t)$ instead of $u_0(t)$

$$\Big(H(t) = \Big)\, u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (*)$$

Heaviside

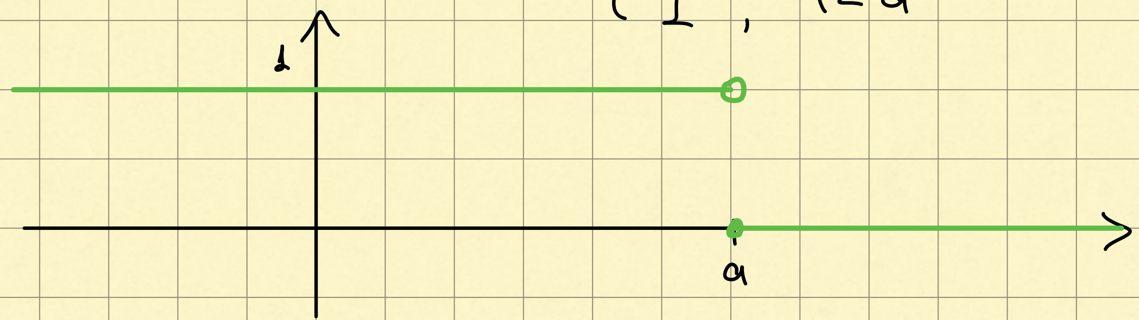
Note: $u_a(t) = u(t-a)$

why: $t \geq a$ : $u_a(t) = 1$ by ★

$t - a \geq 0 \Rightarrow u(t-a) = 1$ by ★

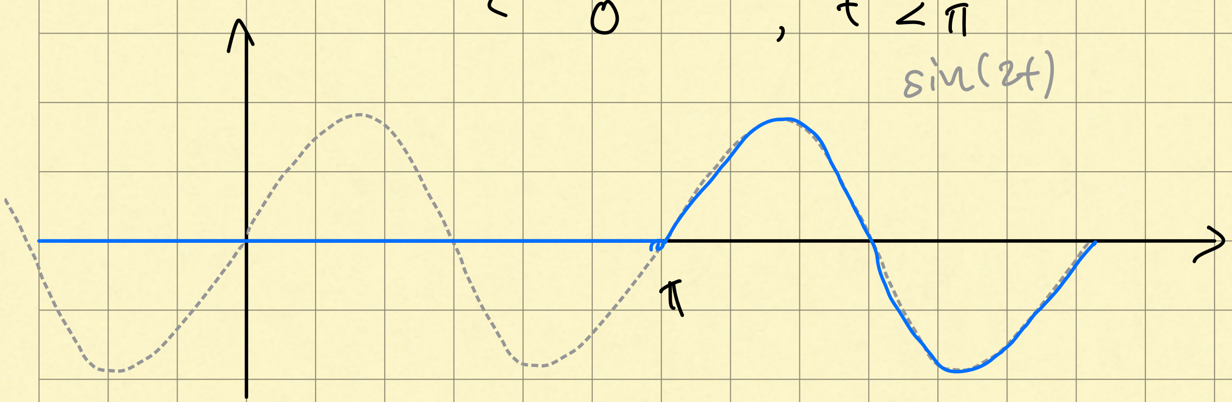
similarly for $t < a$.

Also: $1 - u_a(t) = \begin{cases} 0, & t \geq a \\ 1, & t < a \end{cases}$




Plan: use $u_a$, $1-u_a$ to model signals starting w/ time delay and/or stopping at a certain time.

Ex 1. $f_1(t) = \begin{cases} \sin(2t), & t \geq \pi \\ 0, & t < \pi \end{cases}$

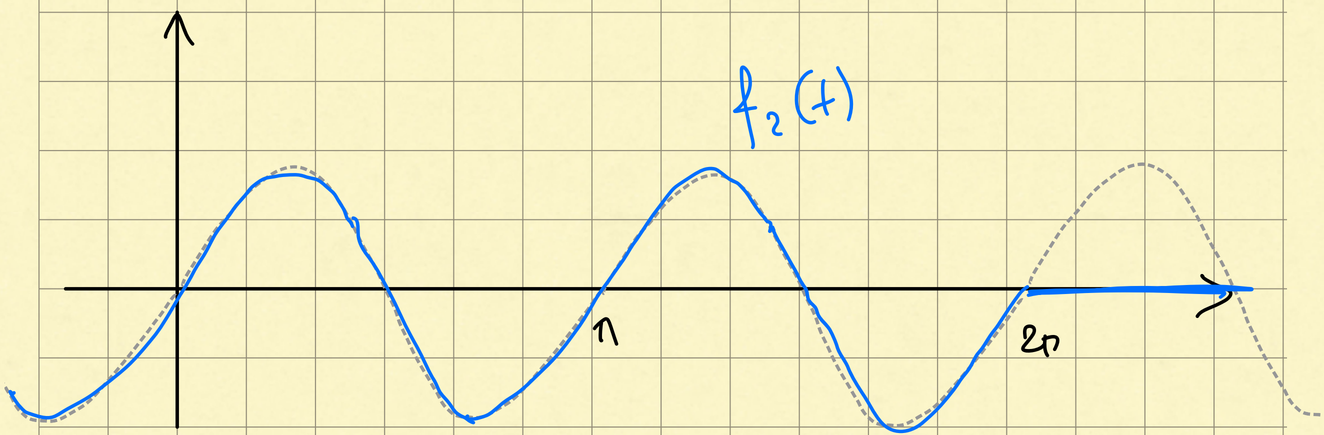


Want: express $f_1(t)$ using step functions.

$$f_1(t) = u_\pi(t)\sin(2t)$$
$$= u(t-\pi)\sin(2t)$$

Ex 2: $f_2(t) = \begin{cases} \sin(2t), & t \le 2\pi \\ 0, & t > 2\pi \end{cases}$

signal stopping at $t = 2\pi$.

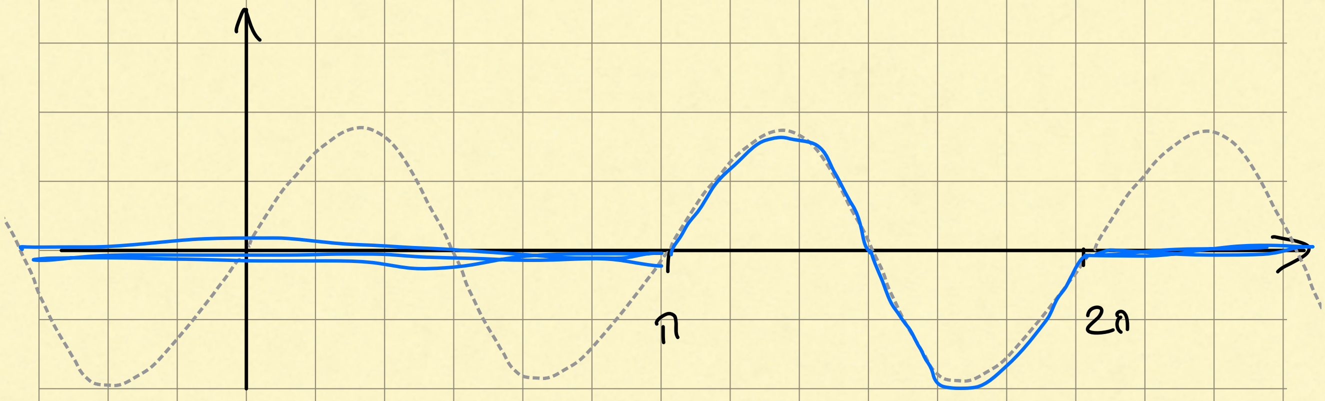



$$f_2(t) = (1 - \underbrace{u_{2\pi}(t)}_{\begin{cases} 0, & t \geq 2\pi \\ 1, & t < 2\pi \end{cases}})\sin(2t)$$

//

Ex 3: $f_3(t) = \begin{cases} \sin(2t) & \pi \le t < 2\pi \\ 0 & t < \pi \text{ or } t \geq 2\pi \end{cases}$

signal w/ time delay which stops at $t = 2\pi$.




$$f_3(t) = u_\pi(t)\, f_2(t)$$

$$= (1 - u_{2\pi}(t))\, f_1(t)$$

$$= u_\pi(t)\,(1 - u_{2\pi}(t))\, \sin(2t)$$

$$= \underbrace{\left(u_\pi(t) - u_{2\pi}(t)\right)}\, \sin(2t)$$

why: $u_\pi(t)\,(1 - u_{2\pi}(t)) =$

$$= u_\pi(t) - u_\pi(t)\, u_{2\pi}(t)$$

$$= u_\pi(t) - u_{2\pi}(t).$$

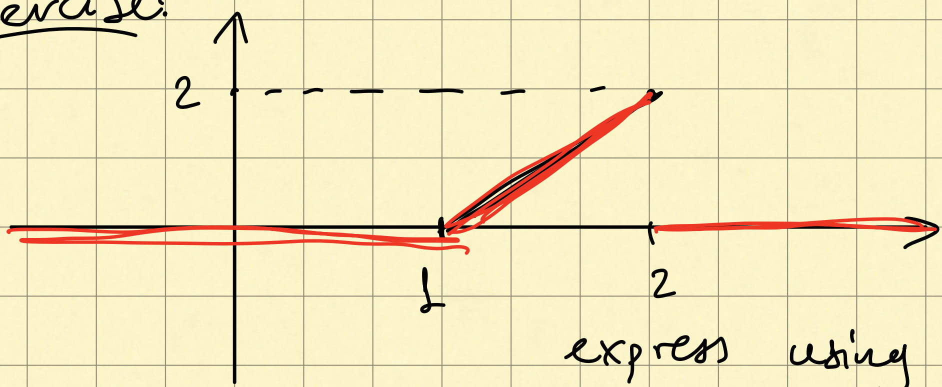
Sol: $f_3(t) = \sin(2t)\left(u_\pi(t) - u_{2\pi}(t)\right)$ //

Why we do this: to take $\mathcal{L}$:

Rule: $\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$\Leftrightarrow$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a)$.

Exercise:




express using step fcts.

Ans: $f(t) = \left(u_1(t) - u_2(t)\right)2(t-1)$