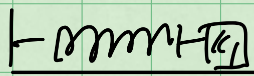
Laplace of piecewise cont. fcts.

last time: 

$$\begin{cases} x'' + 9x = f(t) \\ x(0) = x'(0) = 0 \end{cases} \qquad f(t) = \begin{cases} \sin(2t), & t \in [\pi, 2\pi) \\ 0 & \text{otherwise.} \end{cases}$$

$$X(s) = \mathcal{L}\{x(t)\} = 2\left(e^{-\pi s} - e^{-2\pi s}\right) \frac{1}{s^2+4} \frac{1}{s^2+9}.$$

Goal: find $x(t)$.

Rule: $\mathcal{L}\{u(t-a) f(t-a)\} = e^{-as} \mathcal{L}\{f(t)\}$

$$\Leftrightarrow \quad \mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a) \mathcal{L}^{-1}\{F(s)\}(t-a)$$

$$X(s) = 2e^{-\pi s} \frac{1}{s^2+4} \frac{1}{s^2+9} - e^{-2\pi s} 2 \frac{1}{s^2+4} \frac{1}{s^2+9}$$

$$\Rightarrow x(t) = u(t-\pi)\, \mathcal{L}^{-1}\left\{\frac{2}{s^2+4} \frac{1}{s^2+9}\right\}(t-\pi)$$

$$+ \; u(t-2\pi)\, \mathcal{L}^{-1}\left\{\frac{2}{s^2+4} \frac{1}{s^2+9}\right\}(t-2\pi)$$

$$\frac{2}{s^2+4} \frac{1}{s^2+9} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$= \ldots = \frac{4}{5} \frac{1}{s^2+4} - \frac{4}{5} \frac{1}{s^2+9}$$

for A & B: multiply by (s^2+4), set $s = 2i$, take Real & Imaginary pts.

So: $x(t) = u(t-\pi)\left\{\frac{2}{5}\sin(2t) - \frac{4}{15}\sin(3t)\right\}(t-\pi)$

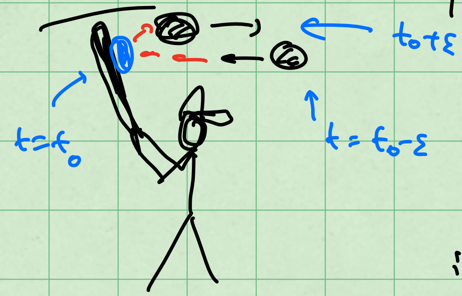
$- u(t-2\pi)\left\{\frac{2}{5}\sin(2t) - \frac{4}{15}\sin(3t)\right\}(t-\pi)$

$\Rightarrow x(t) = u(t-\pi)\left(\frac{2}{5}\sin(2(t-\pi)) - \frac{4}{15}\sin(3(t-\pi))\right)$

$- u(t-2\pi)\left(\frac{2}{5}\sin(2(t-2\pi)) - \frac{4}{15}\sin(3(t-2\pi))\right)$ //

7.6 delta function.

Goal: model forces acting instantaneously.



At time t_0 bat hits the ball
At time $t_0+\varepsilon$ ball moving in opposite direction

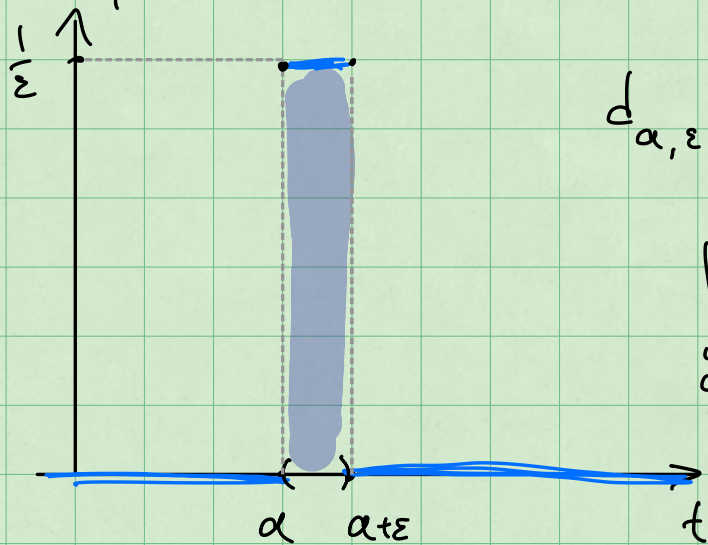
$$\Delta p = p_2\big|_{t=t_0+\varepsilon} - p_1\big|_{t=t_0} = m\underbrace{v_2}_{v(t_0+\varepsilon)} - m\underbrace{v_1}_{v(t_0)}$$

$$\overset{FTC}{=} \int_{t_0}^{t_0+\varepsilon} \frac{d}{dt}(m\overset{\text{velocity}}{v})\,dt$$

$$= \int_{t_0}^{t_0+\varepsilon} \underbrace{f(t)}_{\text{force}}\,dt \quad \Big\} \text{ Impulse of } f \text{ over interval } [t_0, t_0+\varepsilon]$$

Rmk: Δp depends on integral of f (impulse), not on its pointwise values.

Now: set up a simple function w/ impulse 1, over a short time interval, to model a force.



$$d_{a,\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon}, & a \le t \le a+\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_0^\infty d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1.$$

Note: max of $d_{a,\varepsilon} \to \infty$ as $\varepsilon \to 0$.

Want: take limit as $\varepsilon \to 0$ of these functions $d_{a,\varepsilon}$.

Result: we make sense of the Dirac delta function $\delta_a(t)$

Informally/intuitively:

$$\delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t) = \begin{cases} \infty, & t = a \\ 0, & \text{otherwise.} \end{cases}$$

not a rigorous defn.

More rigorously: δ_a is an operator: it eats a function $f(t)$, spits out its value at $t=a$.

continuous

$$f(t) \longrightarrow \boxed{\delta_a(t)} \longrightarrow f(a)$$

Notation

$$\cancel{\left(\delta_a(t)\right)\left(f(t)\right)} = \int_0^\infty f(t)\,\delta_a(t)\,dt = f(a).$$

notation, not true integral.

Ex:

$$\int_0^\infty 1\,\delta_a(t)\,dt = 1 \qquad a \geq 0$$

$$\int_0^\infty \sin(t)\,\delta_{-\frac{\pi}{2}}(t)\,dt = \sin\left(-\frac{\pi}{2}\right) = -1$$

$$\boxed{\int_0^\infty e^{-st}\,\delta_a(t)\,dt = e^{-sa} \qquad s > 0}$$

By def'n, $\mathcal{L}\{\delta_a(t)\} = \int_0^\infty e^{-st}\,\delta_a(t)\,dt = e^{-as}$

Motivation for integral notation in def'n of $\delta_a(t)$

f cont:

$$\int_0^\infty f(t)\, d_{a,\varepsilon}(t)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(t)\,dt$$

$$d_{a,\varepsilon} = \begin{cases} \frac{1}{\varepsilon}, & t\in[a,a+\varepsilon] \\ 0, & \text{otherwise}\end{cases} \qquad \overset{FTC}{=} f(\bar{t}),\quad \bar{t}\in[a,a+\varepsilon]$$

$$\xrightarrow{\varepsilon\to 0} f(a)$$

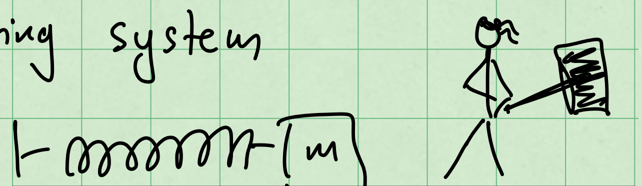
If we could make sense of $\delta_a = \lim d_{a,\varepsilon}$

$$"\int_0^\infty f(t) \lim_{\varepsilon\to 0} d_{a,\varepsilon}\,dt \overset{?}{=} \lim_{\varepsilon\to 0}\int_0^\infty f(t)\,d_{a,\varepsilon}\,dt$$

$$= f(a).$$

not a rigorous computation.

Ex: IVP: mass-spring system



hit mass w/ hammer at $t=3$, impulse $5N$

Initially mass at rest: $x(0) = x'(0) = 0$

$m = 1, \quad k = 4, \quad c = 0$

$$x'' + 4x = \underbrace{f(t)}_{\text{force}}$$

impulse.

$$f(t) = 5\,\delta_3(t)$$

acting at time $t=3$

Take Laplace:

$\mathcal{L}\{5\,\delta_3(t)\}$

$$s^2 X(s) - s\,\cancel{x(0)}^{\,0} - \cancel{x'(0)}^{\,0} + 4X(s) = 5e^{-3s}$$

$$\Rightarrow \quad X(s) = \frac{5e^{-3s}}{s^2+4}$$

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,\mathcal{L}^{-1}\{F(s)\}(t-a)}$$

$$\Rightarrow \quad x(t) = u(t-3)\,\mathcal{L}^{-1}\left\{\frac{5}{s^2+4}\right\}(t-3)$$

$$= u(t-3)\,\frac{5}{2}\sin(2t)\Big|_{t-3}$$

$$= u(t-3)\,\frac{5}{2}\sin(2(t-3)) \quad //$$

disp. becomes $\neq 0$ after $t=3$