OH today 3-5

No OH tomorrow

7.6 Duhamel's formula: it is a formula used for computing the response of a linear system (spring-mass, RLC circuit etc) to any given input.

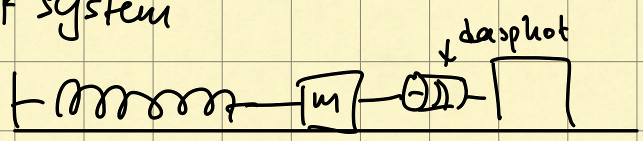
Given spring mass, RLC system described by a 2nd order linear ODE:

$$a x'' + b x' + c x = f(t) \quad (1)$$

parameters of system

external input.

Ex:



$$m x'' + c x' + k x = \underbrace{f(t)}_{\text{external force}}$$

mass

damping

spring const.

Solve (1) w/ initial cond. $x(0) = 0$, $x'(0) = 0$

Take $\mathcal{L}$:

$$a s^2 X(s) + b s X(s) + c X(s) = F(s)$$

$$\Rightarrow X(s) = \frac{1}{a s^2 + b s + c} F(s)$$

Call

$$W(s) = \frac{1}{as^2+bs+c}$$ transfer function (frequency response)

W indep. of input fct $f(t)$, only depends on system.

So: $X(s) = W(s)F(s)$.

convolution!

$$x(t) = w(t) * f(t)$$

$w(t) = \mathcal{L}^{-1}\{W(s)\}$ weight function / impulse response.

So:

$$\boxed{x(t) = \int_0^t w(t-\tau)\, f(\tau)\, d\tau}$$

← Duhamel's formula

If $w(t)$ is known we can predict the output to any given input.

Ex: Last time: spring-mass system

$$\begin{cases} x'' + 4x = f(t), & f(t) = 5\delta_3(t) \\ x(0) = x'(0) = 0 \end{cases}$$

Write Duhamel's formula:

$$s^2 X(s) + 4X(s) = F(s)$$

$$\Rightarrow \quad X(s) = \frac{1}{s^2+4} F(s)$$

transfer function

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+4} \right\} * f(t)$$

$$= \left(\frac{1}{2} \sin(2t) \right) * f(t)$$

impulse response/ weight fct

$$\Rightarrow \quad x(t) = \int_0^t \frac{1}{2} \sin(2(t-\tau)) f(\tau) d\tau.$$

If we plug in $f(\tau) = 5\delta_3(\tau)$:

$$x(t) = \int_0^t \frac{5}{2} \sin(2(t-\tau)) \delta_3(\tau) d\tau$$

$$= \int_0^\infty \Big(\underbrace{u(t-\tau)}_{0 \text{ if } t-\tau<0 \Leftrightarrow \tau>t} \Big) \frac{5}{2} \sin(2(t-\tau)) \delta_3(\tau) d\tau$$

$$= u(t-3) \cdot \frac{5}{2} \sin(2(t-3))$$

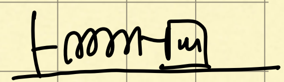
$w(t)$: impulse response: output corr. to input $\delta_0(t)$

Ch 9. Fourier Series.

Write periodic functions as infinite sums of sines & cosines.

Motivation: $x'' + \omega_0^2 x = A\cos(\omega t)$ ← freq. ω

↑ natural freq.



$\omega^2 \neq \omega_0^2$

Undetermined coef. / Laplace transf.:

a particular sol'n

$$x_p = \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t)$$

periodic, freq. ω, period $\frac{2\pi}{\omega}$

If we have external force $\underbrace{\sum_{n=1}^{m} A_n \cos(\omega_n t)}_{\text{sum cosines as external input}}$

a part. sol'n

$$\tilde{x}_p = \sum_{n=1}^{m} \frac{A_n}{\omega_0^2 - \omega_n^2}\cos(\omega_n t) \qquad (\omega_n^2 \neq \omega_0^2)$$

What if we have more general periodic inputs?

Periodic functions

A function $f(t)$, $t \in \mathbb{R}$, is called periodic if there exists $p > 0$ so that

$$f(t+p) = f(t) \quad \text{for all } t \in \mathbb{R}.$$

Such a p is called a period. If there exists a smallest period it is called the period.

Ex:

1. Any constant function is periodic. Any $p > 0$ is a period, there is no smallest period.

2. $\cos(5t)$: $p = \frac{2\pi}{5}$ is a period (the smallest)

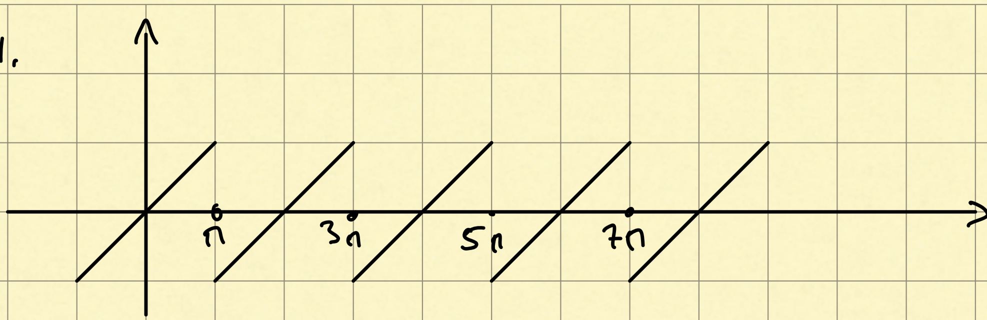
$$\cos(5(t+p)) = \cos(5t + 2\pi) = \cos(5t)$$

Notice: $n \frac{2\pi}{5}$ is a period for any integer n. Ex: $\frac{4\pi}{5}$, $\frac{6\pi}{5}$ are periods.

3. $3\cos(5t) + 2\sin(12t) - \cos(3t) + 2$ periodic, 2π is a period.

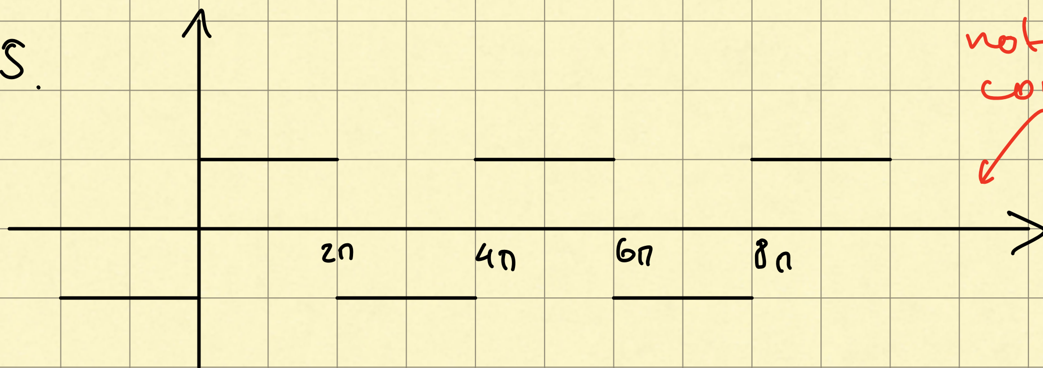
In general: $\cos(nt)$, $\sin(nt)$ are periodic, 2π is a period (n integer)

4.



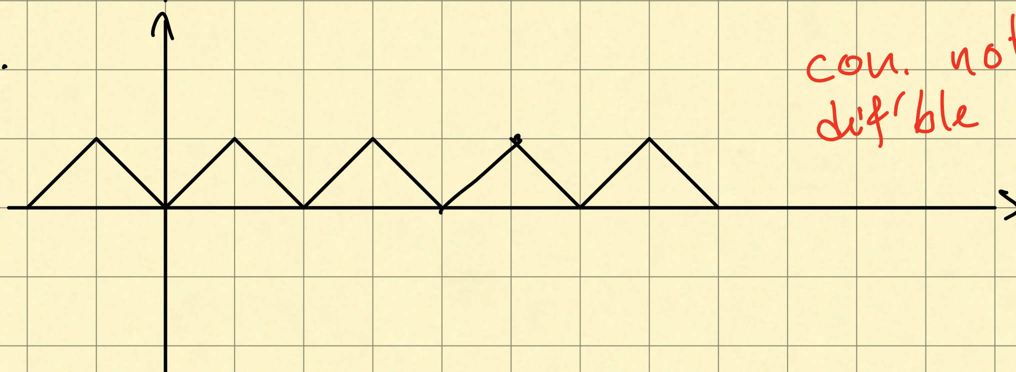
not continuous

5.



6.

con. not dif'ble



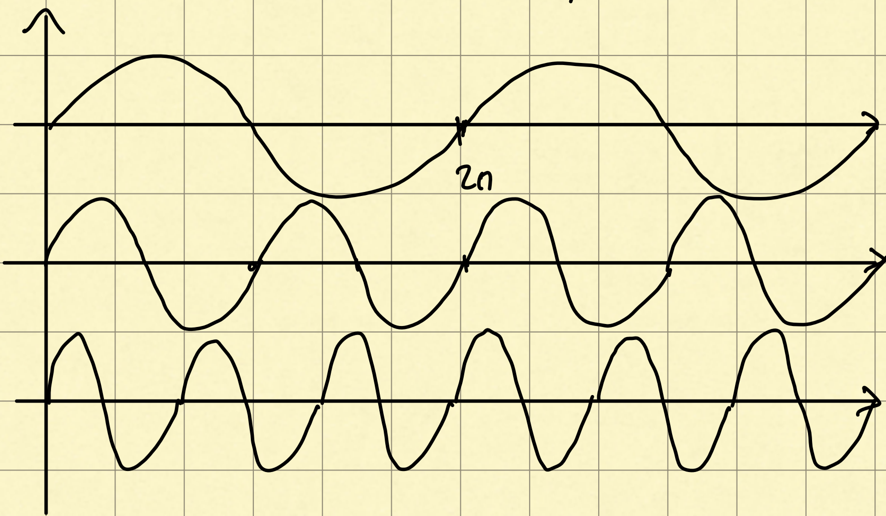
Can't hope to write 4, 5, 6 as finite sums of sines & cosines bec. they are not dif'ble / continuous everywhere.

<u>Fourier's approach:</u> attempt to write a periodic fct of period 2π as an <u>infinite</u> sum of sines & cosines.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\underbrace{nt}) + b_n \sin(\underbrace{nt}) \right)$$

periodic, $\frac{2\pi}{n}$ is a period.

for some const. a_n, b_n.



Under mild assumptions on periodic fct f the series converges to f (at pts where it is continuous).

Next time: compute coefficients a_n, b_n examples.