Plan:

- Existence & Uniqueness of sols to linear systems
- Structure of sols
- Example

2. - Introductory remarks on how to find a solution.

Last time: $\underline{\underline{x}}' = \underset{n \times n}{\underline{\underline{P}}(t)}\, \underline{\underline{x}} + \underline{\underline{f}}(t)$ (1)

Q: Is there a sol'n? (is there $x(t)$ satisfying (1)?)

Existence & Uniqueness

→ Let $\underline{\underline{P}}(t)$ & $\underline{\underline{f}}(t)$ be continuous on an open interval I which contains number a

all entries of $\underline{\underline{P}}(t)$, $\underline{\underline{f}}(t)$ cont. on I.

→ b n×1 column vector.

Then: (1) has exactly one sol'n on all of I which satisfies $\underline{\underline{x}}(a) = \underline{\underline{b}}$.

special about linear systems

Ex: $\begin{cases} x_1' = \sin(t)\, x_1 + \ln(t+1)\, x_2 \\ x_2' = x_1 + \cos(t)\, x_2 + e^t \end{cases}$

$\underline{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{\underline{P}}(t) = \begin{bmatrix} \sin(t) & \ln(t+1) \\ 1 & \cos(t) \end{bmatrix}$, $\underline{\underline{f}}(t) = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$

Given $\underline{\underline{b}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Want to solve

$$\begin{cases} \underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}} + \underline{\underline{f}}(t) \\ \underline{\underline{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{\underline{b}} \end{cases} \quad (2)$$

Largest interval on which $\underline{\underline{P}}(t)$, $\underline{\underline{f}}(t)$ cont. is $I = (-1, \infty)$; so (2) has exactly one sol'n on I.

Structure of sols.

no non-hom. term.

→ If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are sol's to $\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}}$
then $c_1 \underline{\underline{x}}_1(t) + \ldots + c_n \underline{\underline{x}}_n(t)$ is also a sol'n; $c_1, \ldots, c_n$ constant scalars.

[sol's of $\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}}$ form a vector space]

→ can produce more sol's from a few building blocks. What are good building blocks?

Linear independence

Def: The functions $f_1(t), \ldots, f_n(t)$ are lin. independent on an interval I if

$$c_1 f_1(t) + \ldots + c_n f_n(t) = 0 \text{ on } I$$
$$\Rightarrow c_1 = \ldots = c_n = 0$$

Linearly dependent otherwise.

Ex: t, t^2 on $\mathbb{R}$.

Suppose $c_1 t + c_2 t^2 = 0$ for all $t \in \mathbb{R}$

$$\left.\begin{array}{ll} t=1 \Rightarrow & c_1 + c_2 = 0 \\ t=-1 \Rightarrow & -c_1 + c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$$

so lin. indep.

Thm: If $\underline{\underline{x}}_1, \dots, \underline{\underline{x}}_n$ are lin. indep. sol's of $\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}}$ ($n \times n$ matrix) then <u>any</u> sol'n of $\underline{\underline{x}}' = \underline{\underline{P}}\,\underline{\underline{x}}$ is of the form

$$\underline{\underline{x}} = c_1 \underline{\underline{x}}_1(t) + \dots + c_n \underline{\underline{x}}_n(t)$$

for some $c_1, \dots, c_n$ constant scalars.

How to check lin. independence:

$$\underline{\underline{x}}_1 = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \dots, \underline{\underline{x}}_n = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

Wronskian.

$$W(\underline{\underline{x}}_1, \dots, \underline{\underline{x}}_n)(t) = \det \begin{bmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{bmatrix}$$

No derivatives!

<u>To check lin. indep.</u>

If $\underline{\underline{x}}_1, \dots, \underline{\underline{x}}_n$ are sol's of $\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}}$ on an interval I then:

→ If $\underline{\underline{x}}_1, \dots, \underline{\underline{x}}_n$ are linearly dependent <u>then</u>

$W(\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n) = 0$ on all of I.

→ If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are lin. independent then $W(\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n)(t) \neq 0$ for all t in I

Ex: $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$, $\underline{\underline{A}} = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$

Given sol's: $\underline{\underline{x}}_1 = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{\underline{x}}_2 = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Goal: find a sol'n on all of $\mathbb{R}$ w/ $\underline{\underline{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q: Is there such a sol'n? How many?

By Existence & uniqueness there is exactly one soln on all of $\mathbb{R}$: $\underline{\underline{A}} = \underline{\underline{P}}(t)$ const

$\underline{\underline{f}} = \underline{\underline{0}}$

Q: Are $\underline{\underline{x}}_1, \underline{\underline{x}}_2$ good building blocks?

$$W(\underline{\underline{x}}_1, \underline{\underline{x}}_2) = \det\begin{bmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{bmatrix} = -2e^{5t} + e^{5t} = -e^{5t} \neq 0$$

on $\mathbb{R}$,

so $\underline{\underline{x}}_1, \underline{\underline{x}}_2$ lin. indep.

so any sol'n of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ is

$$\underline{\underline{x}}(t) = c_1\,\underline{\underline{x}}_1(t) + c_2\,\underline{\underline{x}}_2(t).$$

Want: $\underline{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\Rightarrow \quad c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} c_1 + c_2 = 1 \\ -c_1 - 2c_2 = 2 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 4 \\ c_2 = -3 \end{cases}$$

So: $\underline{x} = 4 \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} - 3 \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}$ //