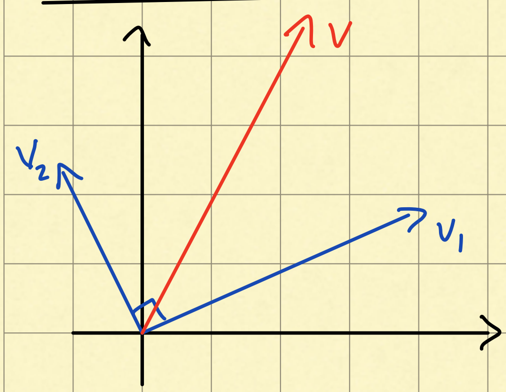
Last time: Hoped to write a 2π-periodic function as infinite sum of trigonometric functions.

$$f \overset{?}{=} \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

Today: find a_n, b_n in terms of f.

Motivation



v_1, v_2 non-zero vectors, orthogonal to each other:

$$v_1 \cdot v_2 = 0$$

v is a given vector, want to write

$$v = a_1 v_1 + a_2 v_2 \quad (1)$$

Want: a_1, a_2. Take dot pr. of (1) w/ v_1:

$$v \cdot v_1 = a_1 \, v_1 \cdot v_1 + a_2 \underbrace{v_2 \cdot v_1}_{0}$$

$$\Rightarrow a_1 = \frac{v \cdot v_1}{v_1 \cdot v_1} = \frac{v \cdot v_1}{|v_1|^2}$$

similarly:

$$a_2 = \frac{v \cdot v_2}{|v_2|^2}.$$

[Here v, v_1, v_2 assumed given
a_1, a_2 unknown] //

Want:

$$f(t) = \frac{a_0}{2} \cdot 1 + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

given

given

unknowns

Think of $f(t)$ as playing the role of v.

$1, \cos(nt), \sin(n)$ — " ——— v_1, v_2

$\int_{-\pi}^{\pi} \cdot \, \cdot \, dt$ — " ——— dot product.

Def'n: 2 functions $u(t), v(t)$ defined on $[a,b]$ are called orthogonal on $[a,b]$ if

$$\int_a^b u(t)\, v(t)\, dt = 0.$$

(think of $\int_a^b \cdot \cdot \, dt$ as being analogous to dot product)

Ex: $u(t) = 1$, $v(t) = \cos(t)$, $[a,b] = [-\pi,\pi]$

$$\int_{-\pi}^{\pi} 1 \cdot \cos(t)\, dt = \sin(t) \Big|_{-\pi}^{\pi} = 0$$

$u(t) = \cos(t)$, $v(t) = \sin(t)$, $[a,b] = [-\pi,\pi]$

$$\int_{-\pi}^{\pi} \cos(t) \sin(t)\, dt \underset{dv = \cos(t)dt}{\overset{u = \sin(t)}{=}} - \int_{-\pi}^{\pi} \cos(t) \sin(t)\, dt = 0$$

or double angle

$$\int_{-\pi}^{\pi} \cos(t)\sin(t)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2t)\,dt = 0.$$

Fact 1 $\quad n, m = 1, 2, 3, \ldots$

a). $$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

b) $$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

c) $$\int_{-\pi}^{\pi} \cos(mt)\sin(nt) = 0 \quad n, m = 1, 2, \ldots, \ldots$$

d) $$\int_{-\pi}^{\pi} \cos(mt)\cdot 1\,dt = \int_{-\pi}^{\pi} \sin(mt)\cdot 1\,dt = 0. //$$

Assumption: f piecewise cont, 2π-periodic has Fourier series, we can integrate it term by term.

$$f(t) = \frac{a_0}{2}\cdot 1 + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

mutually orthogonal

For a_0:

$$\int_{-\pi}^{\pi} 1 \cdot f(t)\,dt = \int_{-\pi}^{\pi} \frac{a_0}{2} \cdot 1\,dt + \sum_{n=1}^{\infty} \Bigg(a_n \underbrace{\int_{-\pi}^{\pi} \cos(nt) \cdot 1\,dt}_{0 \ \text{Fact 1 d}} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nt) \cdot 1\,dt}_{0} \Bigg)$$

$$\Rightarrow \quad a_0 \overbrace{\int_{-\pi}^{\pi} \frac{1}{2}\,dt}^{\pi} = \int_{-\pi}^{\pi} f(t)\,dt$$

$$\Rightarrow \quad \boxed{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt}$$

For a_k, $k \geq 1$: Multiply by $\cos(kt)$

$$\int_{-\pi}^{\pi} f(t) \cos(kt)\,dt = \underbrace{\int_{-\pi}^{\pi} \frac{a_0}{2} \cos(kt)\,dt}_{0 \ \text{Fact 1d.}} + \sum_{n=1}^{\infty} \Bigg(a_n \overbrace{\int_{-\pi}^{\pi} \cos(nt) \cos(kt)\,dt}^{= \begin{cases} \pi, & n=k \\ 0, & n \neq k \end{cases} \ \text{Fact 1a.}} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nt) \cos(kt)\,dt}_{0 \ \text{Fact 1c}} \Bigg)$$

$$\Rightarrow \quad a_k \int_{-\pi}^{\pi} \cos^2(kt)\,dt = \int_{-\pi}^{\pi} f(t) \cos(kt)\,dt$$

Fact 1a $\Rightarrow$ $$\boxed{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt.}$$

Similarly: $$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt$$ (exercise)

<u>Define</u> the Fourier Series of a piecewise cont. function $f(t)$ w/ period 2π as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

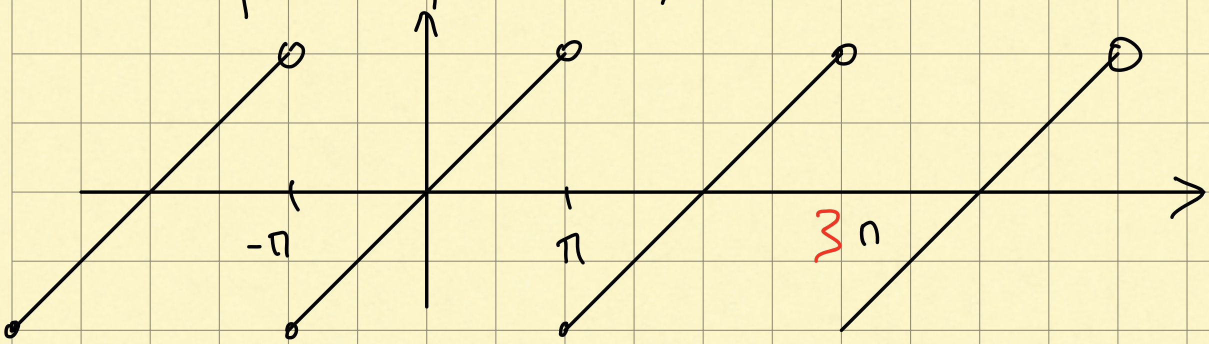
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt.$$

Write: $$\underset{\uparrow}{f} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right).$$

the right hand side is the expansion of f, not claiming it converges to f.

Ex: Compute F.S. of 2π-periodic fct $f(t) = t$, $t \in [-\pi, \pi)$



Find: a_0, a_k, $k = 1, 2, \ldots$