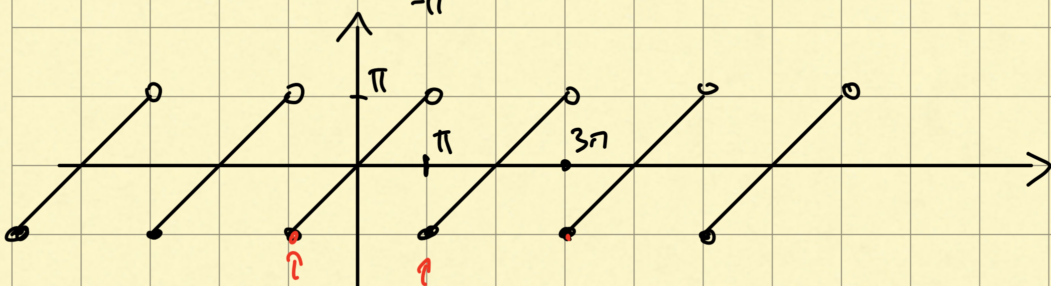
Last time: Fourier Series of 2π-periodic functions

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nt) f(t)\, dt \quad (n \geq 0)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nt) f(t)\, dt \quad (n \geq 1)$$



$f(t) = t, \quad -\pi \leq t < \pi, \quad 2\pi$-periodic.

$a_n \stackrel{?}{=}$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t \, dt = \frac{1}{\pi} \left. \frac{t^2}{2} \right|_{-\pi}^{\pi} = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos(nt)\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t \frac{d}{dt} \left(\frac{\sin(nt)}{n} \right) dt$$

$$= \underbrace{\frac{1}{\pi} \left. t \frac{\sin(nt)}{n} \right|_{-\pi}^{\pi}}_{0} - \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\frac{\sin(nt)}{n}}_{0} dt$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{f}{t}\ \overbrace{\frac{d}{dt}\left(-\frac{\cos(nt)}{n}\right)}^{g'}\,dt$$

$$= \frac{1}{\pi}\ t\left(-\frac{\cos(nt)}{n}\right)\Bigg|_{-\pi}^{\pi} + \cancel{\frac{1}{\pi}\int_{-\pi}^{\pi} 1\cdot\frac{\cos(nt)}{n}\,dt}$$

$$0$$

$$= \frac{1}{\pi}\left(\pi\left(-\frac{\cos(n\pi)}{n}\right) + (-\pi)\frac{\cos(nt)}{n}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\pi}{n}\cos(n\pi)\right) = -\frac{2}{n}\cos(n\pi)$$

$$= \begin{cases} -\frac{2}{n}\cdot 1 & n \text{ even} \\ -\frac{2}{n}\cdot(-1) & n \text{ odd} \end{cases} = -\frac{2}{n}(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$f \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nt) \qquad (a_n = 0 \text{ for all } n)$$

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More general periodic functions

$f(t)$ w/ period $P = 2L$ (L half period)

Want: F.S. for $f(t)$.

Consider: $g(u) = f(\overbrace{\tfrac{L}{\pi}u}^{t})$

$$g(u+2\pi) = f\left(\frac{L}{\pi}(u+2\pi)\right) =$$

$$= f\left(\frac{Lu}{\pi} + \overbrace{2L}^{P}\right) = f\left(\frac{Lu}{\pi} + P\right)$$

$$= f\left(\frac{Lu}{\pi}\right) = g(u).$$

So: g is 2π-periodic.

$$g(u) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(nu) + b_n \sin(nu)\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos(nu)\,du, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin(nu)\,du.$$

$$f(t) \overset{t=\frac{\pi}{L}u}{=} g\left(\frac{\pi}{L}u\right)$$

$$\Rightarrow \boxed{f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)\right)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}u\right)\cos(nu)\,du$$

$$\overset{t=\frac{L}{\pi}u}{=} \quad \frac{1}{\pi}\cdot\frac{\pi}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt.}$$

Note: if $L = \pi$ we find original formula.

Similarly:

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt}$$

Boxes: F.S. for $2L$-periodic functions.