<u>Last time:</u> Fourier Series for functions of period $P = 2L$.

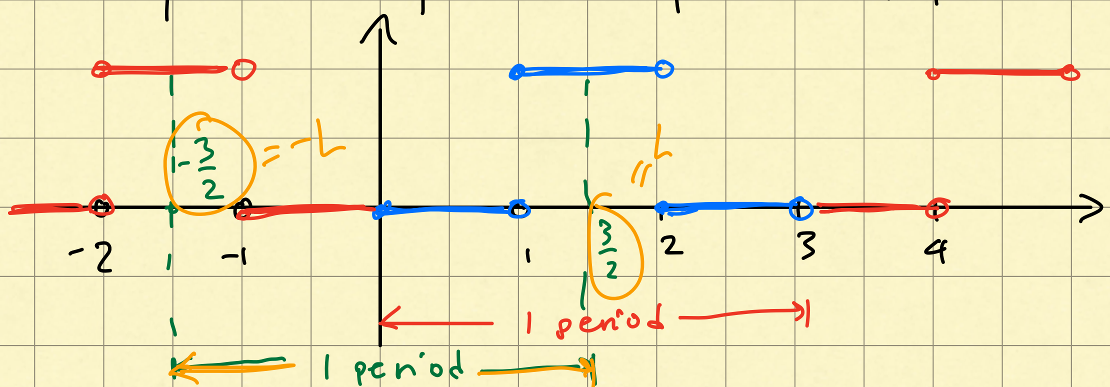
$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} t\right) + b_n \sin\left(\frac{\pi n}{L} t\right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{\pi n}{L} t\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{\pi n}{L} t\right) dt$$

<u>Ex:</u> $f(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & 0 \le t < 1, \quad 2 \le t < 3 \end{cases}$

$f(t+3) = f(t)$ ( periodic w/ period 3).




$L = \frac{3}{2}$

<u>Observation:</u> If $f$ periodic of period $2L$ then

$$a_n = \frac{1}{L} \int_{0}^{2L} f(t) \cos\left(\frac{n\pi}{L} t\right) dt$$

$$b_n = \frac{1}{L} \int_{0}^{2L} f(t) \sin\left(\frac{n\pi}{L} t\right) dt$$

$a_n = ?$ $\quad$ $f(t) = 1$ for $1 \le t < 2$

$$a_0 = \frac{1}{3/2} \int_1^2 1\, dt = \frac{2}{3}$$

$$a_n = \frac{1}{3/2} \int_1^2 1 \cos\left(\frac{2\pi}{3} n t\right) dt$$

$$= \frac{2}{3} \frac{3}{2\pi n} \sin\left(\frac{2\pi}{3} n t\right) \Big|_1^2$$

$$= \frac{1}{\pi n} \Big[ \underbrace{\sin\left(\frac{4\pi}{3} n\right) - \sin\left(\frac{2\pi}{3} n\right)}_{= \tilde{a}_n} \Big]$$

↑ repeats itself whenever 3 is added to n.

Note:

$$\tilde{a}_{n+3} = \sin\left(\frac{4\pi}{3}(n+3)\right) - \sin\left(\frac{2\pi}{3}(n+3)\right)$$

$$= \sin\left(\frac{4\pi}{3} n + 4\pi\right) - \sin\left(\frac{2\pi}{3} n + 2\pi\right)$$

$$= \sin\left(\frac{4\pi}{3} n\right) - \sin\left(\frac{2\pi}{3} n\right) = \tilde{a}_n$$

$n = 1$:

$$\tilde{a}_n = \sin\left(\frac{4\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$n=2$: $\tilde{a}_n = \sin\left(\frac{8\pi}{3}\right) - \sin\left(\frac{4\pi}{3}\right) =$

$$= \sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{4\pi}{3}\right) = \sqrt{3}$$

$n=3$: $\tilde{a}_n = \sin(4\pi) - \sin(2\pi) = 0.$

So:
$$a_n = \begin{cases} \frac{1}{\pi n}(-\sqrt{3}) & n = 3k+1 \\ \frac{1}{\pi n}(\sqrt{3}) & n = 3k+2 \\ 0 & n = 3k+3 \end{cases} \qquad k \geq 0.$$

$$a_0 = \frac{2}{3}$$

For the $b_n$:

$$b_n = \frac{1}{3/2} \int_1^2 \sin\left(\frac{\pi n}{3/2} t\right) dt$$

$$= \frac{1}{3/2} \frac{3/2}{\pi n} \left(-\cos\left(\frac{\pi n}{3/2}\right)\right)\Bigg|_1^2$$

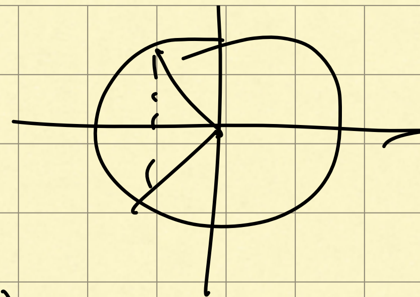
$$= \frac{1}{\pi n} \left(\underbrace{\cos\left(\frac{2}{3}\pi n\right) - \cos\left(\frac{4}{3}\pi n\right)}_{=: \tilde{b}_n}\right)$$

Again: $\tilde{b}_{3+n} = \tilde{b}_n$

$n=1$: $\cos\left(\frac{2}{3}\pi\right) - \cos\left(\frac{4}{3}\pi\right) = 0$

$n=2$ : $\cos\left(\frac{4}{3}\pi\right) - \cos\left(\frac{8\pi}{3}\right)$

$$= \cos\left(\frac{4}{3}\pi\right) - \cos\left(\frac{2\pi}{3}\right)$$
$$= 0$$



$n=3$ : $\cos\left(\frac{6\pi}{3}\right) - \cos\left(\frac{12\pi}{3}\right) = \cos(0) - \cos(0)$
$$= 0.$$

So: $b_n = 0$ for all $n \geq 1$. //.

## Convergence of Fourier Series

When does the F.S. of a periodic fct $f$ converge to $f$?

↳ want: for each fixed $t$:

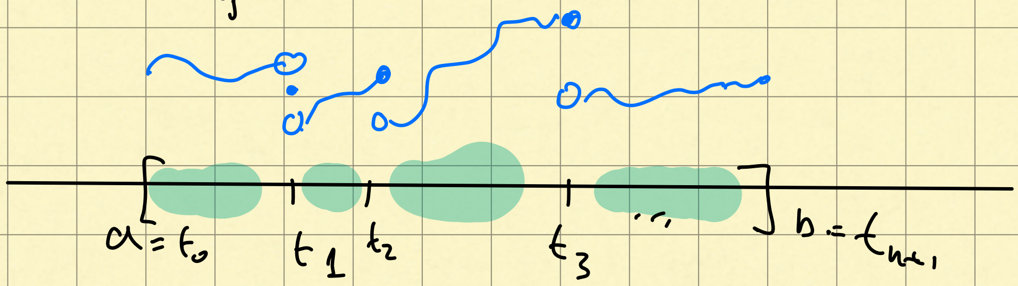
$$\lim_{N\to\infty}\left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n\cos\left(\frac{n\pi}{L}t\right) + b_n\sin\left(\frac{n\pi}{L}t\right)\right)$$
$$= f(t).$$

[might not converge "equally fast" for all $t$].

Def: $f$ piecewise cont. on $[a,b]$ if there are $a = t_0, t_1, \dots, t_{n+1} = b$ where $t_j \in [a,b]$ such that:

- $f$ cont. on $(t_{j-1}, t_j)$

- $\lim_{t \to t_j^{\pm}} f(t)$ exists & is finite.




Non-example:

$$f(t) = \begin{cases} 1, & t = 0 \\ \frac{1}{t}, & 0 < t \leq 1 \end{cases}$$

$$\lim_{t \to 0^+} f(t) = \infty.$$