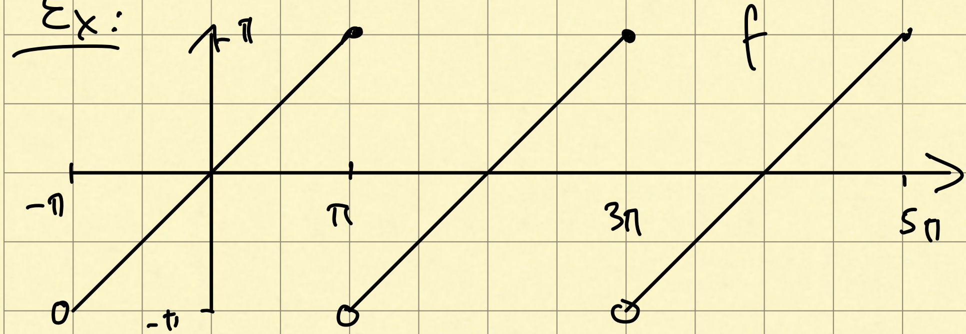
Convergence of F.S.

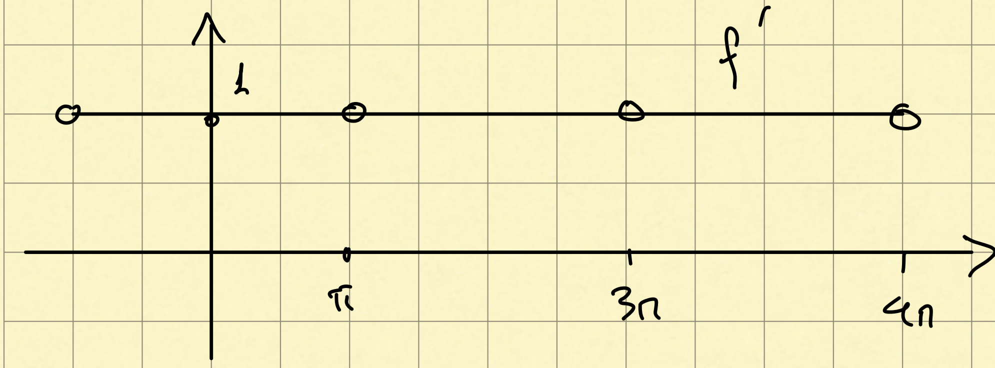
Defined: piecewise cont. functions.

A function f defined on an interval $[a,b]$ is <u>piecewise</u> <u>smooth</u> if it is piecewise cont. & its derivative (defined away from points of discontinuity of f) is also piecewise continuous.

Ex:



piecewise cont.



derivative undefined at $\pi, 3\pi, 5\pi$ ...

f' piecewise cont.

$f \to$ piecewise smooth.

Theorem: If f periodic & piecewise smoth then its F. S. converges

a) to $f(t)$ for all t where f continuous

b) to the average

$$\frac{1}{2}\left(f(t^+) + f(t^-)\right)$$

at t's for which f is discont.

Here

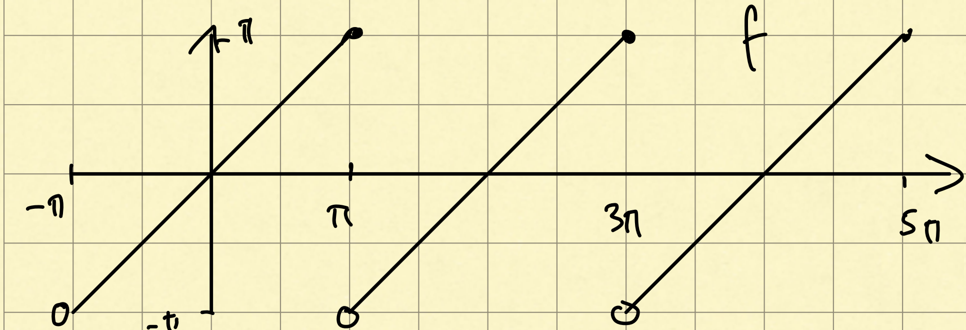
$$f(t^+) = \lim_{s \to t^+} f(s).$$

Note: if f cont. at t: $\frac{1}{2}\left(f(t^+) + f(t^-)\right)$

$$= f(t)$$

Ex: $f(t) = t, \quad -\pi \leq t < \pi$

$f(t + 2\pi) = f(t)$



theorem applies.

$$f \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(n\ t)$$

Ex:

When $t=0$:

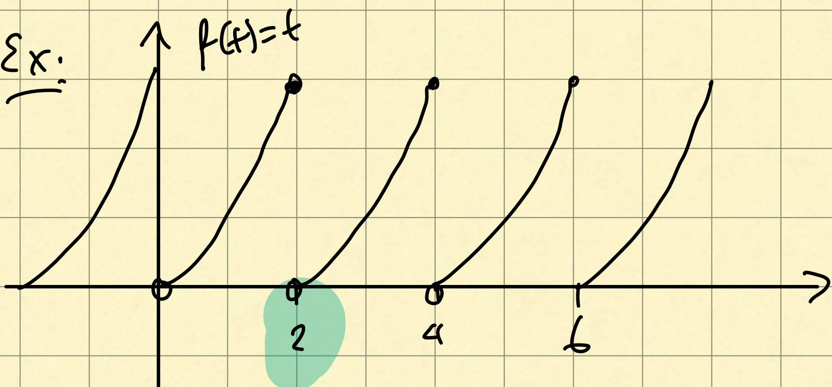
$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \cancel{\sin(0)}^{\,0} = 0 = f(0)$$

$t=\pi$:

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \cancel{\sin(n\cdot\pi)}_{0} = 0 = \frac{1}{2}\left(\lim_{t\to\pi^-} f(t) + \lim_{t\to\pi^+} f(t)\right)$$

$$= \frac{1}{2}\left(\pi + (-\pi)\right) = 0$$

Ex.



periodic, period 2

$L = 1$

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\pi t) + b_n \sin(n\pi t)\right)$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(t)\,dt = \int_0^2 t^2\,dt = \frac{t^3}{3}\Big|_0^2 = \frac{8}{3}$$

$$a_n = \frac{4}{\pi^2 n^2}, \quad b_n = -\frac{4}{\pi n} \qquad \text{(check!)}$$

Plug in $t=2$, into the series.

$$\frac{4}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{\pi^2 n^2}\underbrace{\cos(n\pi\cdot 2)}_{=1} + \left(-\frac{4}{\pi n}\right)\cancelto{0}{\sin(n\pi\cdot 2)}\right)$$

$$= \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \underset{\text{theorem}}{=} \frac{1}{2}\left(f(2^+) + f(2^-)\right)$$

$$= \frac{1}{2}(0+4) = 2$$

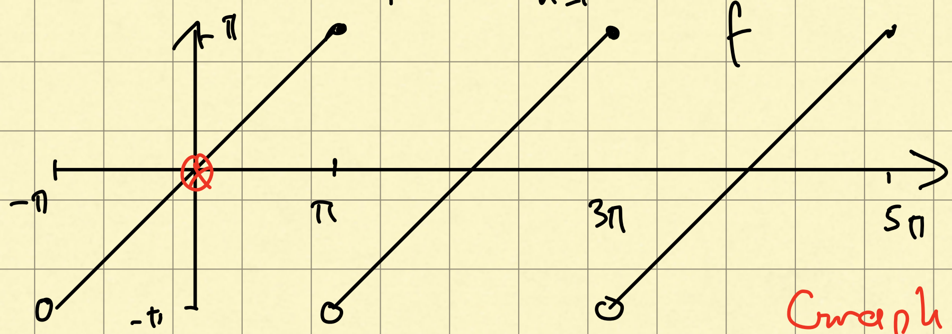
$$\Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{4}{3} = 2 \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad //$$

Exercise: plug in $t=1$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ //

Odd & Even functions

Found: $f \sim \displaystyle\sum_{n=1}^{\infty} b_n \sin(nt)$ ($a_n = 0$ for all n)

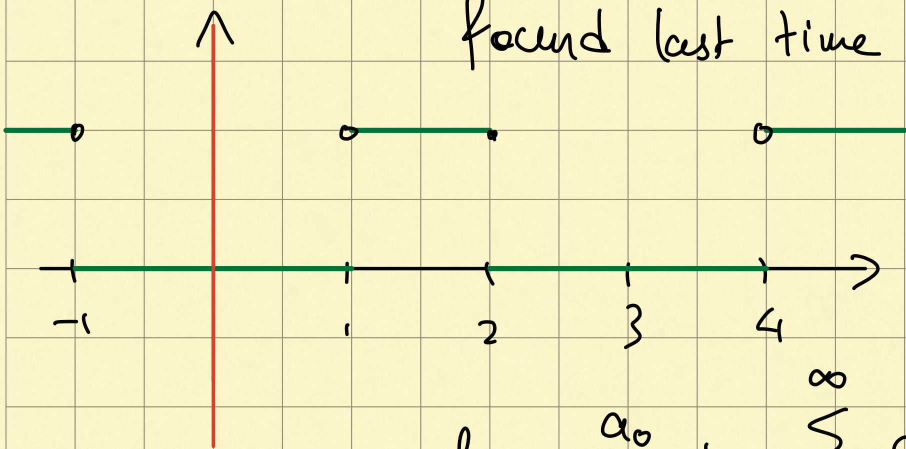


1

Graph symmetric about origin.

Also:

found last time



Graph sym. about y axis

(2)

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{3} n t\right)$$

(all b_n's were 0).

Def'n: → $f(t)$ is even if $f(t) = f(-t)$ for all t. Ex: (2)

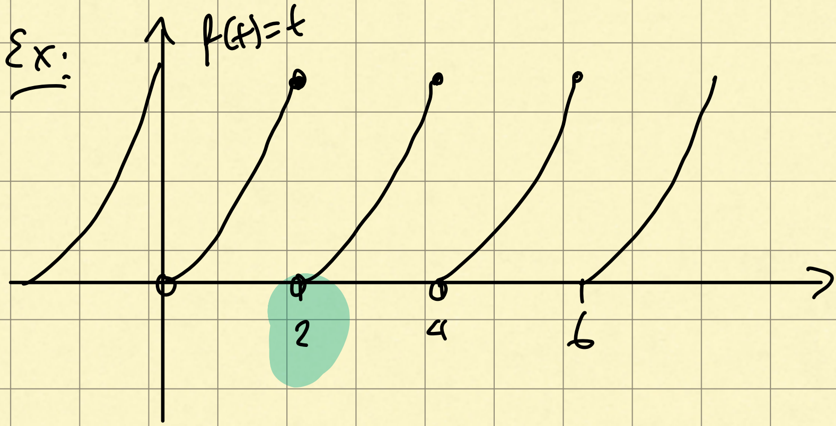
or: $\cos(t) = \cos(-t)$

→ $f(t)$ is odd if $f(t) = -f(-t)$ for all t.

Ex: (1), $\sin(t) = -\sin(-t)$

Note: a function need not be odd or even:

Ex: $f(t) = t^2$



Note: if f even

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt$$

$$\overset{s=-t}{=} -\int_{a}^{0} \underbrace{f(-s)}\,ds + \int_{0}^{a} f(t)\,dt$$

$$\overset{f \text{ even}}{=} \int_{0}^{a} f(s)\,ds + \int_{0}^{a} f(t)\,dt$$

$$= 2\int_{0}^{a} f(t)\,dt.$$

If f odd: $\displaystyle\int_{-a}^{a} f(t)\,dt = 0.$

Now: f periodic, period $2L$, f even

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{\pi n}{L}t\right) + b_n \sin\left(\frac{\pi n}{L}t\right)\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{\pi n}{L}t\right)dt = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{\pi n}{L}t\right)dt$$

even × even = even

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{\pi n}{L}t\right)dt = 0$$

even × odd = odd

So: if f is even & periodic:

no sine terms!

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right)$$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{\pi n}{L} t\right) dt \quad ''$$