Last time:

→ If f even $(f(t) = f(-t)$ for all $t)$ and $2L$-periodic, F.S. has only cosine terms ($b_n = 0$ for all n)

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right)$$

$$a_n = \frac{2}{L} \int_0^L \overbrace{f(t) \cos\left(\frac{n\pi}{L} t\right)}^{\text{even.}} dt$$

→ If f is odd $(f(t) = -f(-t)$ for all $t)$ and $2L$-periodic, F.S. has only sine terms. ($a_n = 0$ for $n = 0, 1, \ldots$)

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right)$$

$$b_n = \frac{2}{L} \int_0^L \underbrace{f(t) \sin\left(\frac{n\pi}{L} t\right)}_{\text{even if}} dt \quad f(t) \text{ is odd.}$$

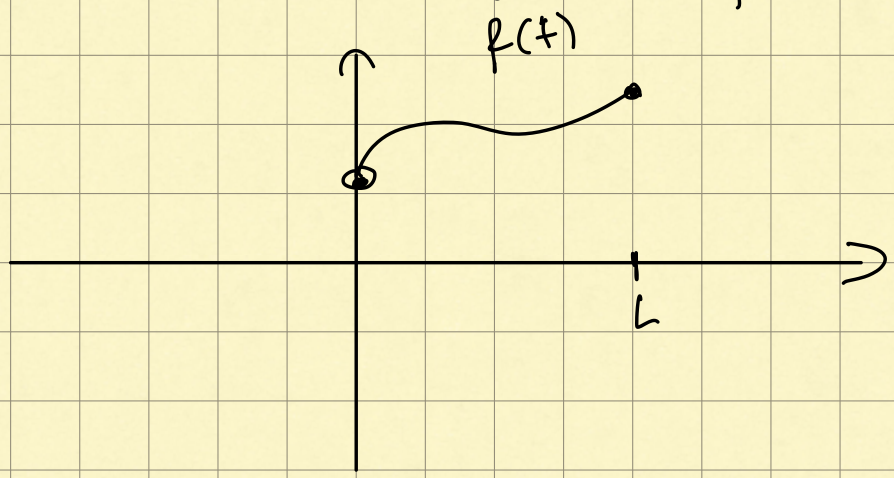
Check: $\begin{cases} \text{even} \times \text{even} = \text{even} \\ \text{even} \times \text{odd} = \text{odd} \\ \text{odd} \times \text{odd} = \text{even} \end{cases}$

Extensions, Fourier sine & Cosine series.

Given: $f(t)$ piecewise cont., defined on $[0, L]$ (no periodicity assumed)

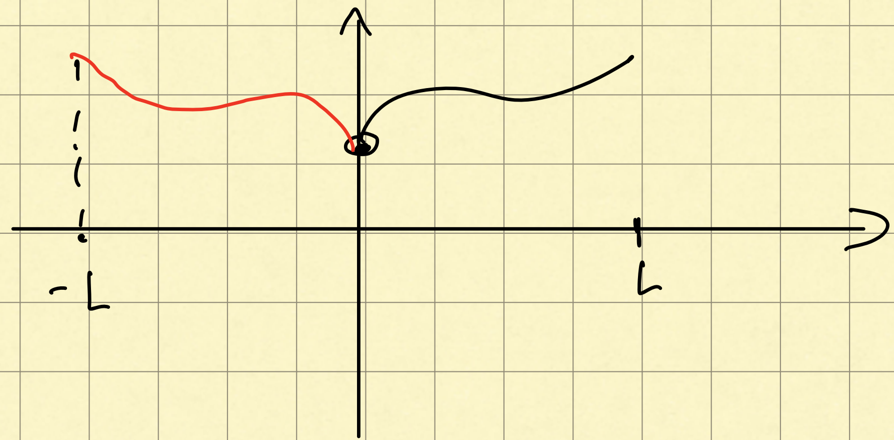
Want: use F.S. to analyze.
Need: periodic function.

What we'll do: extend f to be periodic.



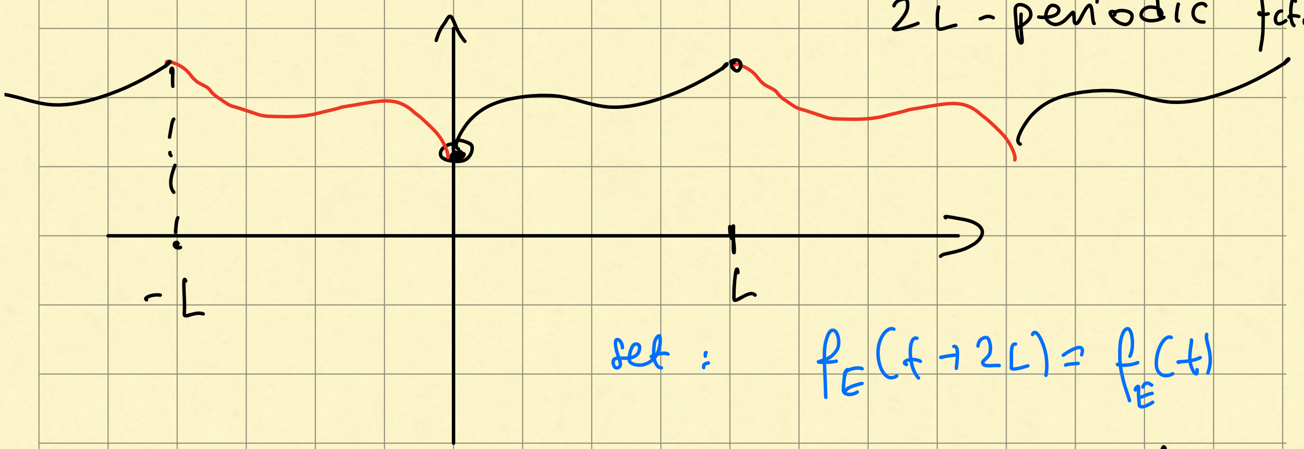
1st step: Extend f to $[-L, L]$. 2 natural choices.

Choice 1: Even extension.



$$f_E(t) = \begin{cases} f(t), & t \in [0, L] \\ f(-t), & t \in [-L, 0] \end{cases}$$

Step 2: Extend to all of $\mathbb{R}$ as 2L-periodic fct.



set: $f_E(t+2L) = f_E(t)$

At points of discontinuity: can define f as the average of the side limits.

Now: f_E 2L-periodic, even $\Rightarrow$ F.S.!

So:

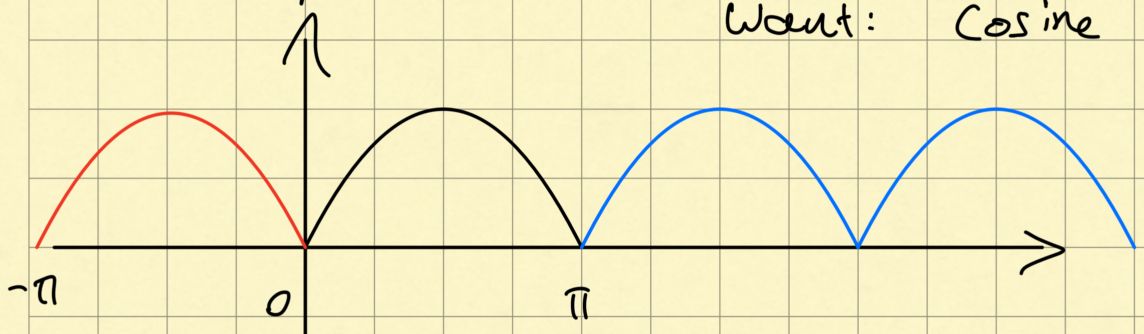
Fourier Cosine Series of f.

$$\begin{cases} f_E \sim \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) \\ a_n = \dfrac{2}{L} \displaystyle\int_0^L f(t) \cos\left(\frac{\pi n}{L} t\right) dt \end{cases}$$

If f piecewise smooth, series converges to f on $[0,L]$ at pts where f continuous & to average of side limits otherwise.

Ex: $f(t) = \sin(t)$, $t \in [0, \pi]$

Want: Cosine F.S.



Even extension:

$$f_E(t) = \begin{cases} \sin(t) & t \in [0,\pi] \\ \sin(-t) & t \in [-\pi, 0] \end{cases}$$

$$= |\sin(t)|$$

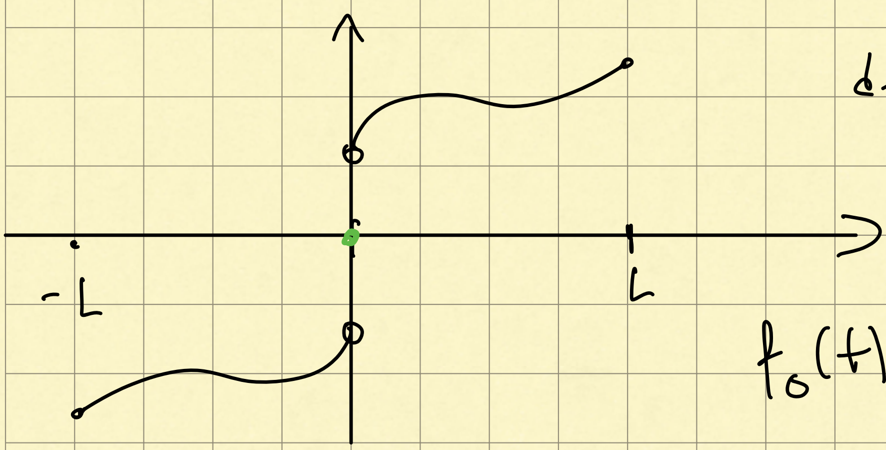
F. cosine series:

$$f_E \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{\sin(t)\cos(nt)\,dt}_{\text{compute in HW}}.$$

<u>Back to Step 1:</u> extend f from $[0,L]$ to $[-L,L]$

2nd choice: odd extension.



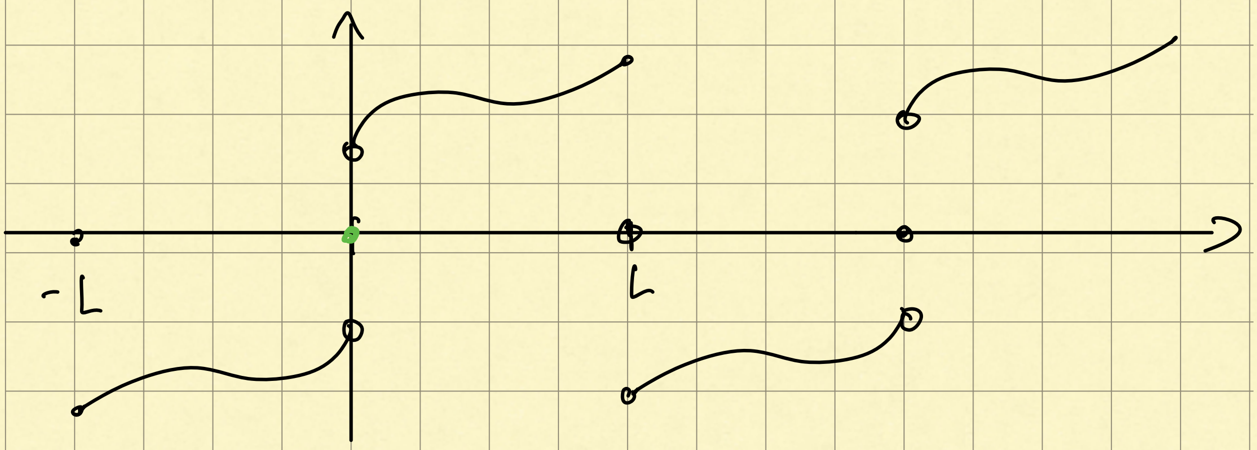
define

$$f_o(t) = \begin{cases} f(t), & t \in (0,L] \\ -f(-t), & t \in [-L,0) \\ 0, & t = 0 \end{cases}$$

can define f_o to have the average value of side limits at points of discont.

Step 2: extend to $\mathbb{R}$:

(at discontinuities we can define f as average of side limits)



Take F.S.

Fourier sine series.

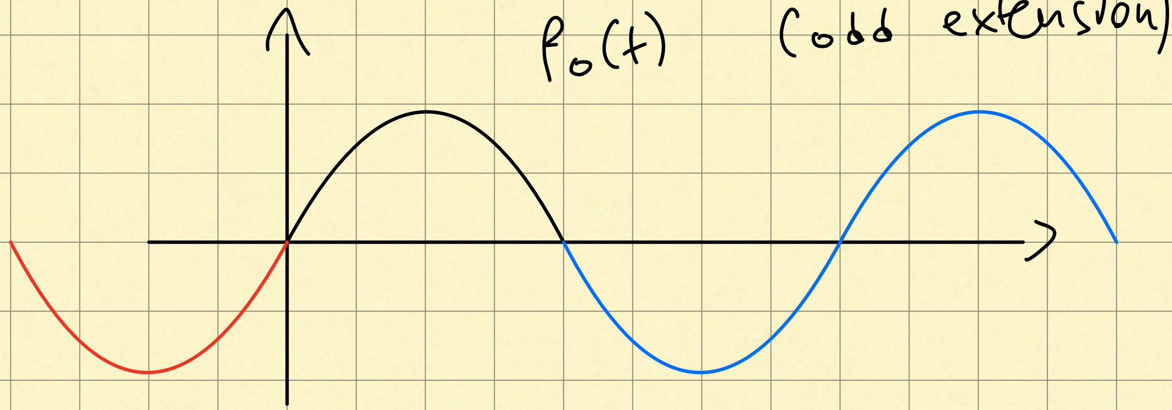
$$f_o \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right)$$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt$$

If f piecewise smooth on $[0, L]$ sine F.S. converges to f (at points where f cont.)

Rmk: Fourier sine & cosine series converge to different functions outside of $[0, L]$

Ex: $f(t) = \sin(t)$, $t \in [0, \pi]$



F. sine series:

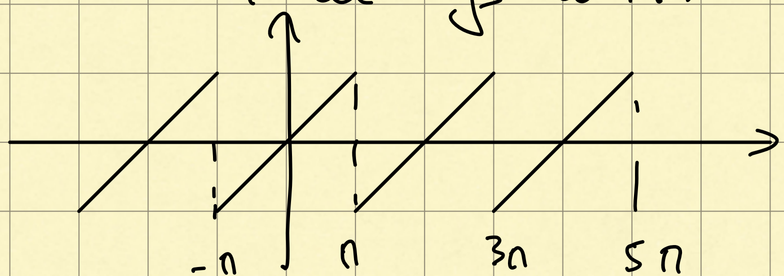
$$f_0(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$= \sin(t) \qquad \begin{pmatrix} b_n = 0 \\ n \neq 1 \\ b_1 = 1 \end{pmatrix}$$

Fourier series & differentiation

Hope: differentiate F.S. term by term

Issue: doesn't always work.



On $(-\pi, \pi)$

$$t = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nt)$$

take term-by-term derivative:

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\cancel{n}} \cancel{n} \cos(nt)$$

Plug in $t=0$: $\sum_{n=1}^{\infty} 2(-1)^{n+1}$ doesn't conv.

even though

$f(t) = t$ nice near 0