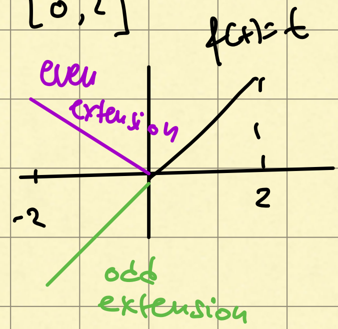
Last time: solve boundary value problems using Fourier Series.

① $\begin{cases} a x'' + b x' + c x = f(t) & \text{on } [0,L] \\ x(0) = x(L) = 0 \end{cases}$

② $\begin{cases} a x'' + b x' + c x = f(t) & \text{on } [0,L] \\ x'(0) = x'(L) = 0 \end{cases}$

Today: $b = 0$ (no x' term)

Last time: $\begin{cases} x'' + 2x = t & \text{on } [0,2] \\ x(0) = x(2) = 0 \end{cases}$



Made observation: If we use an odd extension for $f(t) = t$ (compute F. sine series for f) then the sol'n x will have no cosine terms. If we use a cosine series for $f(t) = t$ (i.e. use even extension), then the sol'n x has no sine terms. ! This uses the fact that there is no x' term in LHS.

Exercise: $\begin{cases} x'' + 2x = t & \text{on } [0,2] \\ x'(0) = x'(2) = 0. \end{cases}$

1. Sine or cosine series for $f(t) = t$?

   Cosine series.

2. Find cosine series for f .

Reminder: $f(t)$ on $[0,L]$ the cosine series:

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right)$$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi}{L} t\right)$$

3. Find series for x (x will have cosine series, as we observed)

Cosine series for f:

$$a_0 = \frac{2}{2} \int_0^2 t\, dt = \left.\frac{t^2}{2}\right|_0^2 = 2$$

$$a_n = \frac{2}{2} \int_0^2 t \cos\left(\frac{n\pi}{2} t\right) dt$$

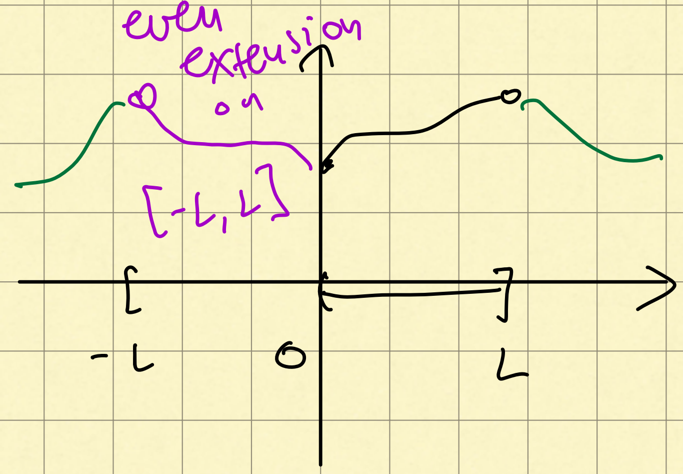
$$= \int_0^2 t \, \frac{2}{\pi n} \frac{d}{dt} \sin\left(\frac{n\pi}{2} t\right) dt$$

$$= \overbrace{\left.\frac{2t}{\pi n} \sin\left(\frac{n\pi}{2} t\right)\right|_0^2}^{0} - \int_0^2 \frac{2}{\pi n} \sin\left(\frac{n\pi}{2} t\right) dt$$

$$= + \int_0^2 \left(\frac{2}{\pi n}\right)^2 \frac{d}{dt} \cos\left(\frac{n\pi}{2} t\right) dt$$

$$= \frac{4}{\pi^2 n^2}\left(\cos(\pi n) - 1\right)$$

$$= \frac{4}{\pi^2 n^2}\left((-1)^n - 1\right)$$



<u>Find x:</u> x will have cosine series

$$x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\underbrace{2}_{=L}} t\right)$$

Want to plug into $x'' + 2x = t$ to find A_n.

$$x'' = \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{2}\right)^2 (-1) \cos\left(\frac{n\pi}{2} t\right)$$

$$x'' + 2x = t$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(A_n\left(\frac{n\pi}{2}\right)^2(-1) + 2A_n\right)\cos\left(\frac{n\pi}{2}t\right)$$

$$+ 2\frac{A_0}{2} = \overset{a_0}{\frac{2}{2}} + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left((-1)^n-1\right)\cos\left(\frac{n\pi}{2}t\right)$$

Match terms:

$$\begin{cases} A_n\left[\left(\frac{n\pi}{2}\right)^2(-1) + 2\right] = \frac{4}{\pi^2 n^2}\left((-1)^n - 1\right) \\ A_0 = 1 \end{cases}$$

$$\Rightarrow A_n = \frac{4}{\pi^2 n^2}\,\frac{(-1)^n - 1}{2 - \left(\frac{\pi n}{2}\right)^2}$$

So:

$$x = \frac{1}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n}{2}t\right)$$

Note: $$x'(t) = -\sum_{n=1}^{\infty} A_n\left(\frac{\pi n}{2}\right)\sin\left(\frac{\pi n}{2}t\right)$$

$$\Rightarrow x'(0) = x'(2) = 0.$$

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Termwise integration of F.S.

f $2L$-periodic, piecewise cont.
(not piecewise smooth, i.e. f' need not be piecewise cont.)

Then

$$(1) \quad f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n}{L} t\right) + b_n \sin\left(\frac{\pi n}{2} t\right) \right)$$

which may or may not converge.

Then (1) can be integrated term by term:

$$\int_0^t f(s)\,ds = \int_0^t \frac{a_0}{2}\,ds + \sum_{n=1}^{\infty} \int_0^t a_n \cos\left(\frac{\pi n}{L} s\right) + b_n \sin\left(\frac{\pi n}{2} s\right) ds$$

not a F.S. if $a_0 \neq 0$

$$= \frac{a_0}{2} t + \sum_{n=1}^{\infty} \frac{L}{\pi n} \left(a_n \sin\left(\frac{\pi n}{L} t\right) - b_n \left(\cos\left(\frac{n\pi}{L} t\right) - 1 \right) \right)$$

and this series converges for all t.