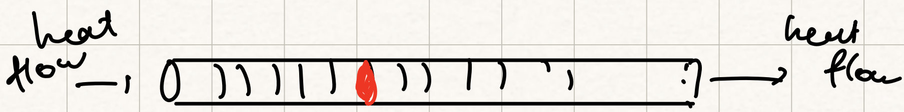
Heat conduction (9.5)

Use Fourier series to study heat conduction on a heated rod.

Given: thin rod w/ cross-section of area A



lateral side insulated

Want to model: temperature u of particles in the rod as time evolves.

Assume: u is const. on each cross-section of the rod.

So: $u = u(x, t)$

time

position on x-axis

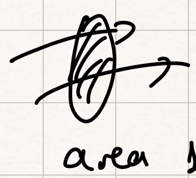
Want to derive

$$\partial_t u = k \, \partial_x^2 u$$

$$\left(\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \right)$$

Partial differential equation. (heat equation in 1 space-dimension)

Heat flux: $\varphi(x,t)$ rate of flow of heat across a cross-section of area 1.

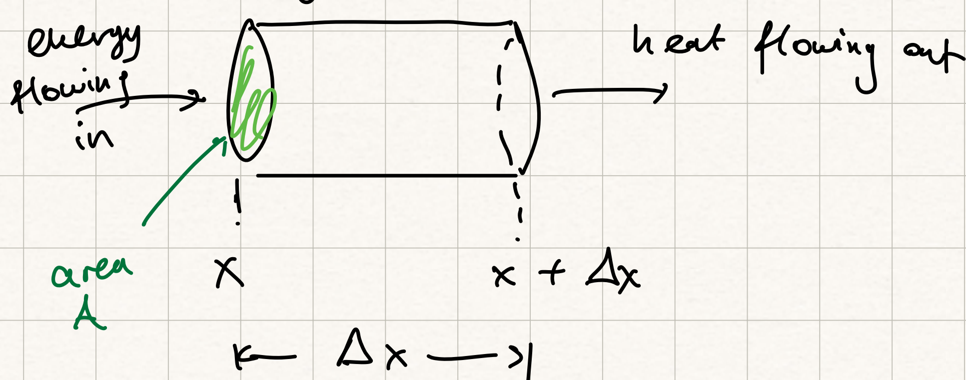


Empirically: $\varphi = -K\,\partial_x u$

energy flows from areas of high temperature to areas of low temperature

K: positive constant (thermal conductivity)

Look at segment



Study rate of heat flow R in segment.

R: calories/s. Strategy: write R in two ways.

1st: $R = A\underbrace{\varphi(x,t)}_{\substack{\text{flux at unit area}\\ \text{cross-section}}} - A\varphi(x+\Delta x,t)$

$= KA\left(\partial_x u(x+\Delta x,t) - \partial_x u(x,t)\right)$

2nd: $R = \dfrac{dQ}{dt}$, where Q is the heat content of segment.

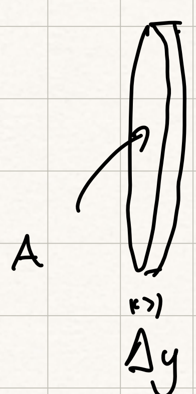
$Q(x,t)$: heat needed to raise temperature from 0 to $u(x,t)$ degrees.

Want to describe Q.

c: specific heat: amount of heat needed to raise temp 1 gram by 1 degree

δ: density of rod

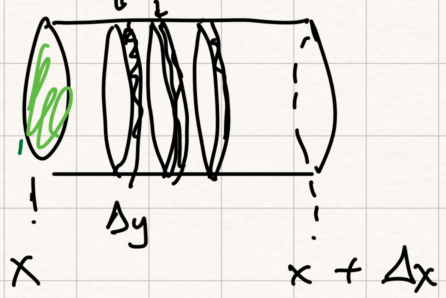
$c\delta$: amount of heat needed to raise temp. of $1\,cm^3$ by 1 deg.



$$\underbrace{(c\delta)\underbrace{(A\,\Delta y)}_{\text{volume}}}_{\text{heat needed to raise temp of slice by 1 deg.}}$$

$$\underbrace{Q = \int_{x}^{x+\Delta x} c\delta\, u(y,t) \cdot A\, dy}_{\text{total heat content of segment.}}$$

heat needed to raise temp. of each slice



add them together to find total heat content.

So:
$$\frac{dQ}{dt} = \int_x^{x+\Delta x} c\delta \, \partial_t u(y,t) \, A \, dy$$

$$R = \frac{dQ}{dt}$$

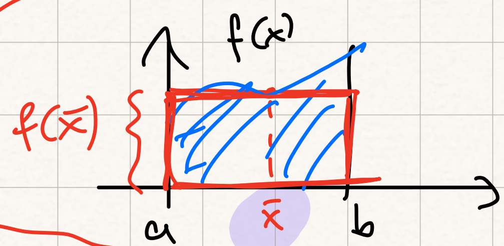
$$\Rightarrow KA\left(\partial_x u(x+\Delta x,t) - \partial_x u(x,t)\right) = \int_x^{x+\Delta x} c\delta \, \partial_t u(y,t) A \, dy$$

|| intermediate v. theorem

$$K\cancel{A}\left(\partial_x u(x+\Delta x,t) - \partial_x u(x,t)\right) = c\delta \, \partial_t u(\bar{x},t) \cancel{A} \underbrace{\Delta x}_{\text{length of interval}}$$

$x \leq \bar{x} \leq x + \Delta x$

$$\Rightarrow \partial_t u(\bar{x},t) = \frac{K}{c\delta} \frac{\partial_x u(x+\Delta x,t) - \partial_x u(x,t)}{\Delta x}$$



$$\int_a^b f(x)dx = f(\bar{x})(b-a)$$

$a < \bar{x} < b$

$\Delta x \to 0$: Since $x \le \bar{x} \le x + \Delta x$

$$\partial_t u(x,t) = \left(\frac{K}{c\delta}\right) \partial_x^2 u(x,t)$$

k thermal diffusivity

So $\boxed{\partial_t u = k\, \partial_x^2 u}$ heat eqn in 1 space dimension.

Boundary conditions.

Recall: $x'' + 5x = 3$

we were given some additional info

such as $x(0) = 0$, $x'(0) = 0$

or $x(0) = 0$, $x(3) = 0$

$x'(0) = 0$, $x'(3) = 0$

we were given this information which helped pinpoint a specific solution of the dif'l eqn from infinitely many possible ones.

In our case now: initial condition will be a function of the spatial variable

$$u(x,0) = f(x)$$

If we have a rod of finite length L:

$0 < x < L$

$f(x)$ describes initial state of system.

Ex: initially temperature of a rod of length 5 is 25 °C

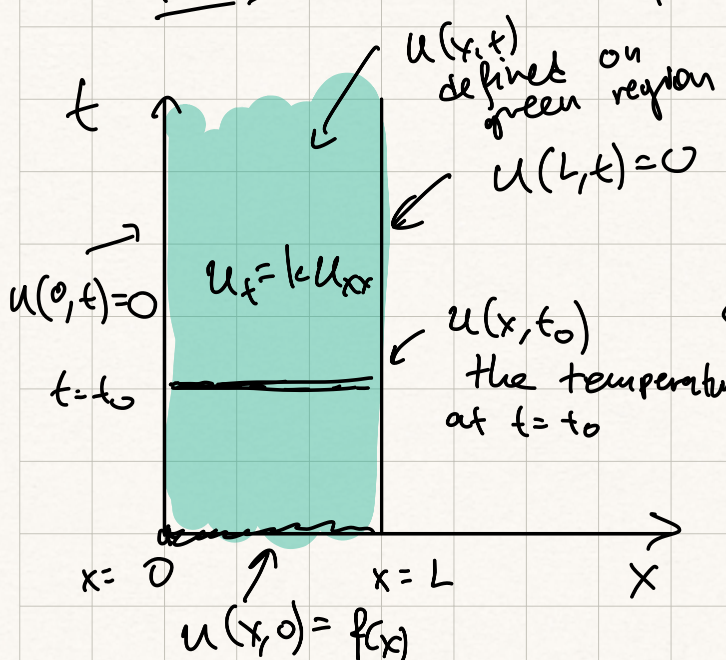
$$u(x,0) = 25 \qquad 0 < x < 5$$

Also: endpoint conditions.

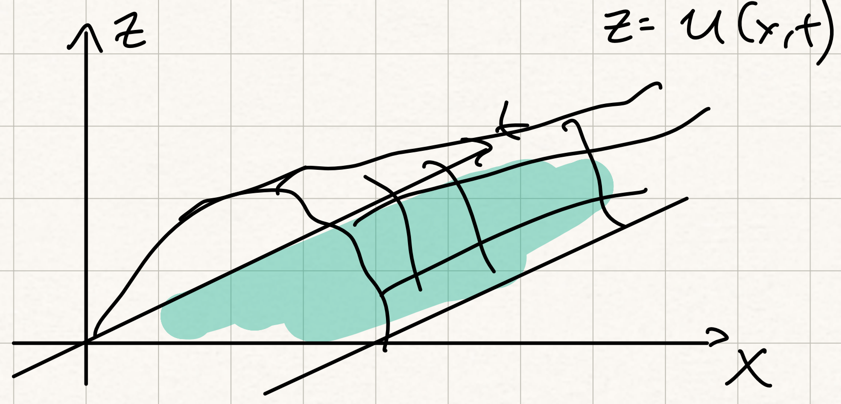
Ex: fix temperature of endpts.

$$u(0,t) = u(L,t) = 0$$

fix temperature to be 0 at endpts for all $t > 0$.



$$\begin{cases} \partial_t u = k\,\partial_x^2 u & 0<x<L,\ t>0 \\ u(x,0) = f(x) & 0<x<L \\ u(0,t) = u(L,t) = 0 & t>0 \end{cases}$$


z
z= u(x,t)
x