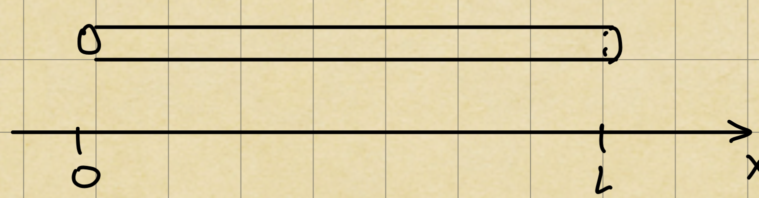
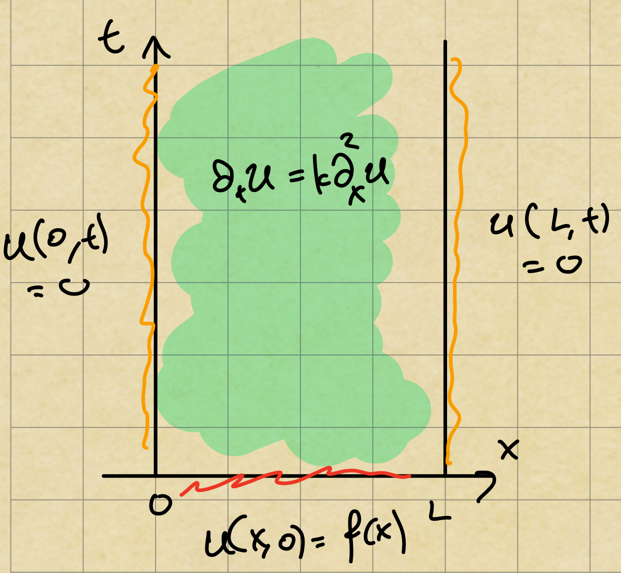
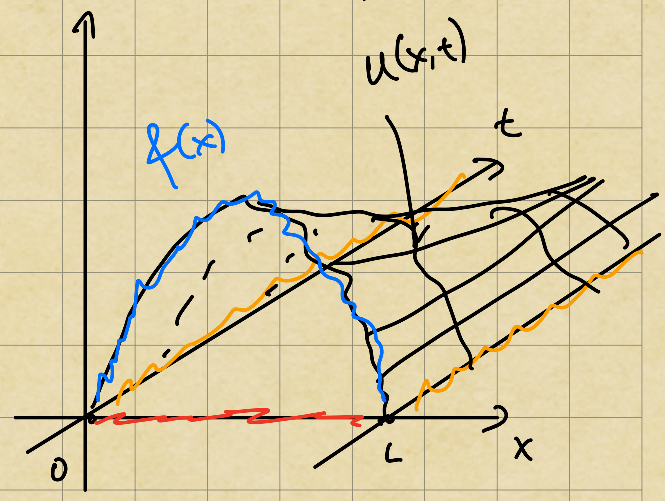
Heat conduction

Thin rod of length L.
Study temperature $u(x,t)$ ↳ time



$$\begin{cases} ① & \partial_t u = k\,\partial_x^2 u & 0<x<L, \quad t>0 \\ ② & u(x,0) = f(x) & 0<x<L \quad \text{initial condition} \\ ③ & u(0,t) = u(L,t) = 0 & t>0 \quad \text{endpoint condition. (fix temperature at endpts)} \end{cases}$$





Endpt condition ③ is called homogeneous (means it's 0)
Initial condition ② is called non-homogeneous

(means it's not 0)

Observations: - If u_1, u_2 solve $\partial_t u = k\partial_x^2 u$ then $c_1 u_1 + c_2 u_2$ also does (c_1, c_2 const.)

- If u_1, u_2 satisfy $u(0,t) = u(L,t) = 0$ then so does $c_1 u_1 + c_2 u_2$

! If $u_1(x,0) = f(x)$, $u_2(x,0) = f(x)$

$u_1(x,0) + u_2(x,0) = 2f(x)$

But if $u_1(x,0) = f_1(x)$, $u_2(x,0) = f_2(x)$ then $c_1 u_1(x,0) + c_2 u_2(x,0) = c_1 f_1 + c_2 f_2$

Strategy: find building blocks u_j, $j = 0, 1, 2, \ldots$ so that

- $\partial_t u_j = k\partial_x^2 u_j \qquad 0 < x < L,\ t > 0$
- $u_j(0,t) = u_j(L,t) = 0 \qquad t > 0$

and form infinite sum

$$u(x,t) = \sum_{j=0}^{\infty} c_j u_j(x,t)$$

where c_j are determined later, so that $u(x,0) = f(x)$.

If f piecewise smooth, infinite sum converges to a sol'n, the sol'n is unique.

Now find building blocks.

Separation of variables. Educated guess:

$$u_j(x,t) = \underbrace{X(x)}_{\text{function of } x}\ \underbrace{T(t)}_{\text{function of } t.}$$

Want: $\partial_t u_j = k\, \partial_x^2 u_j$

$$\Rightarrow \quad X(x)\, T'(t) = k\, X''(x)\, T(t)$$

$$\Rightarrow \quad \underbrace{\frac{T'(t)}{kT(t)}}_{\text{depends only on } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends only on } x}$$

$f(t) = g(x)$
$\partial_t f = 0$
$\Rightarrow f(t) \equiv c$
$\partial_x g = 0$
$\Rightarrow g(x) = c$

$$\Rightarrow \quad \frac{T'}{kT},\ \frac{X''}{X} \text{ constant.}$$

$$\frac{T'}{kT} = \underset{\text{const.}}{\underset{\downarrow}{-\lambda}} \qquad \frac{X''}{X} = -\lambda$$

$$\Rightarrow \quad T' = -\lambda k T \quad , \quad X'' = -\lambda X$$

Recall: $u_j(0,t) = u_j(L,t) = 0$

$\Rightarrow X(0)T(t) = X(L)T(t) = 0$

$T(t) \not\equiv 0 \Rightarrow$ $X(0) = X(L) = 0$

<u>Look</u> at: $\begin{cases} X'' = -\lambda X \\ X(0) = X(L) = 0 \end{cases}$ and seek non-trivial solutions.

If $\lambda = 0$:

$X'' = 0 \Rightarrow X(x) = Ax + B$

$X(0) = 0 \Rightarrow B = 0$

$X(L) = 0 \Rightarrow AL = 0 \Rightarrow A = 0$

so no non-trivial sol's.

If $\lambda = -\alpha^2 < 0$

$X'' = \alpha^2 X$ sols: $e^{\alpha x}, e^{-\alpha x}$

<u>or</u> $X(x) = A\cosh(\alpha x) + B\sinh(\alpha x)$

$\Rightarrow X(0) = 0 \Rightarrow A = 0$

$X(L) = 0 \Rightarrow B\underbrace{\sinh(\alpha L)}_{\neq 0} = 0 \Rightarrow B = 0$

no non-trivial sols.

If $\lambda = \alpha^2 > 0$

$$X'' + \alpha^2 X = 0$$

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow B\sin(\alpha L) = 0$$

want $B \neq 0$, want $\sin(\alpha L) = 0$

What should α be?

want: $\alpha L = n\pi$, n integer

$$\Rightarrow \alpha = \frac{n\pi}{L} \Rightarrow \lambda = \frac{n^2\pi^2}{L^2}.$$

Summarize: If $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ (and only then)

there are non-trivial sols to $\begin{cases} X'' = -\lambda_n X \\ X(0) = X(L) = 0 \end{cases}$

of the form $B\sin\left(\frac{n\pi}{L}x\right)$.

Now,

$$T' = -k\lambda_n T \quad \text{for such } \lambda_n$$

$$\Rightarrow T(t) = C_n e^{-k\lambda_n t} = C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Building blocks: $u_n(x,t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$

Form infinite sum:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

Want: $u(x,0) = f(x)$ (given initial condition)

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \overset{?}{=} f(x)$$

So:

$$\underbrace{c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}_{\text{coef. of sine series of } f.}$$

Summarize: To solve problem

$$\begin{cases} \partial_t u = k\,\partial_x^2 u & 0<x<L,\ t>0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

take:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right),$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Ex: $\begin{cases} \partial_t u = \partial_x^2 u & 0 < x < 5 \quad t > 0 \\ u(0,t) = u(5,t) = 0 \\ u(x,0) = 25 \end{cases}$

Fourier sine series for $f(x) = 25$ on $[0,5]$

$b_n = 50 \dfrac{(-1)^n - 1}{\pi n}$ (check)

So:

$$u(x,t) = \sum_{n=1}^{\infty} 50 \frac{(-1)^n - 1}{\pi n} \underbrace{e^{-\left(\frac{n\pi}{5}\right)^2 t} \sin\left(\frac{n\pi}{5} x\right)}_{\longrightarrow 0 \text{ as } t \to 0}$$