Heat conduction

$$\begin{cases} u_t = k u_{xx} & 0<x<L, \quad 0<t \\ u(0,t) = u(L,t) = 0 & 0<t \\ u(x,0) = f(x) & 0<x<L \end{cases}$$

Derived sol'n:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{L^2}kt} \sin\left(\frac{n\pi x}{L}\right)$$

Observed: $u(x,t) \xrightarrow{t\to\infty} 0$

Today: Insulated endpts.

Rate of heat flow: $\varphi = -k\partial_x u$. If endpts are insulated, no heat coming in & out.

$$\begin{cases} u_t = k u_{xx} & 0<x<L, \ t>0 \\ u_x(0,t) = u_x(L,t) = 0 & t>0 \\ u(x,0) = f(x) & 0<x<L \end{cases}$$

Sol'n: Use separation of variables, find

1 $$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}kt} \cos\left(\frac{n\pi}{L}x\right)$$

Want: $u(x,0) = f(x)$

$$\Rightarrow \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) = f(x)$$

take: (2) $\begin{cases} a_0 = \frac{2}{L}\int_0^L f(x)\,dx \\ a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \end{cases}$

So: sol'n is (1) w/ coef. (2).

Ex: $\begin{cases} 5u_t = u_{xx} & 0<x<10, \quad t>0 \\ u_x(0,t) = u_x(10,t) = 0 & t>0 \\ u(x,0) = 4x & 0<x<10. \end{cases}$

$L = 10$
$k = \frac{1}{5}$

Find coef: $a_0 = \frac{2}{10}\int_0^{10} 4x\,dx = \ldots = 40$

$a_n = \frac{2}{10}\int_0^{10} 4x\cos\left(\frac{n\pi}{10}x\right)dx \overset{IBP}{=} \ldots$

$= \frac{80}{n^2\pi^2}\left((-1)^n - 1\right)$

Plug into (1)

$$u(x,t) = 20 + \sum_{n=1}^{\infty} \frac{80}{n^2\pi^2}\left((-1)^n - 1\right) e^{-\frac{n^2\pi^2}{100}\cdot\frac{1}{5}t}\cos\left(\frac{n\pi}{10}x\right)$$

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HW: book gives you material, look at p. 604 for table containing k for various materials.

Note: 1 : $t \to \infty$

$$u(x,t) \to \frac{a_0}{2}$$

$$= \underbrace{\frac{1}{L}\int_0^L f(x)\,dx}_{\text{average of initial temperature.}}$$

Vibrating string

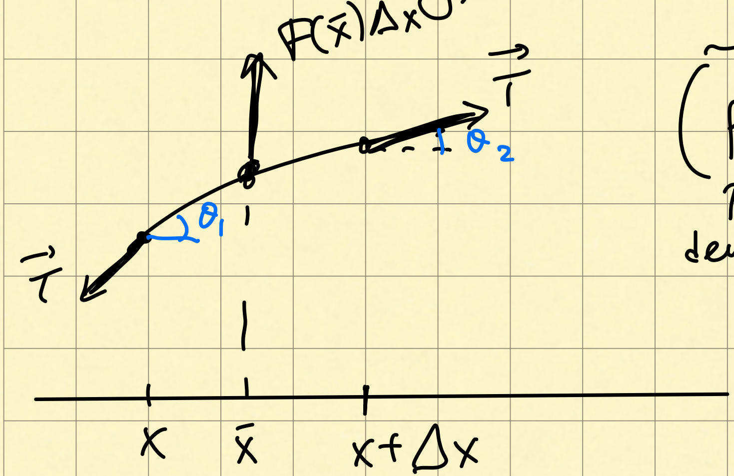
Describe motion of particles. Assume they only move in direction of y axis:

$y(x,t)$: displacement of particle at x at time t.

Linear density: ρ (in $\frac{g}{cm}$)

(maybe) external force of density $F(x)$ $\left(\frac{N}{cm}\right)$
Tension T at endpts. (N)

Zoom into string, use Newton's law



$$\underbrace{\left(\rho \Delta x\right)}_{\text{mass of segment}} y_{tt}(\bar{x},t)$$

ρ: density

$$= F(\bar{x})\Delta x + T\sin(\theta_2) - T\sin(\theta_1)$$

$$\rho\, y_{tt}(\bar{x},t) = F(\bar{x}) + T\,\frac{\sin(\theta_2)-\sin(\theta_1)}{\Delta x}$$

if θ_1, θ_2 very small

$$\approx F(\bar{x}) + T\,\frac{\tan(\theta_2)-\tan(\theta_1)}{\Delta x}$$

$$\approx F(\bar{x}) + T\,\frac{y_x(x+\Delta x,t)-y_x(x,t)}{\Delta x}$$

For small θ:
$\tan\theta \approx \sin(\theta)$
both have value
0 at $\theta=0$ &
derivative 1 at $\theta=0$

Take $\Delta x \to 0$.

$$\boxed{\rho\, y_{tt}(x,t) = F(x) + T y_{xx}(x,t)}$$

If $F(x)=0$

$$\boxed{y_{tt}(x,t) = a^2 y_{xx}(x,t), \quad a^2 = \frac{T}{\rho}}$$

wave eqn in 1 dimension.

Endpoint conditions: string of length L

$$y(0,t) = y(L,t) = 0 \quad \text{(fixed endpts)}$$

Initial conditions: Difference from heat eqn: there one initial condition was enough to specify a soln.

Now we need two:

$$y(x,0) = f(x), \quad y_t(x,0) = g(x)$$

$0<x<L$.

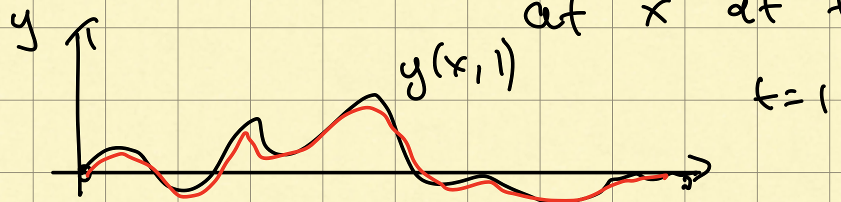
Problem:

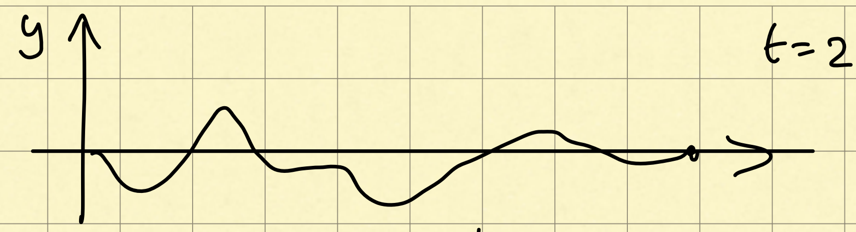
$$\begin{cases} y_{tt} = a^2 y_{xx} & 0<x<L, \ t>0 \\ y(0,t) = y(L,t) = 0 & t>0 \\ y(x,0) = f(x) & 0<x<L \\ y_t(x,0) = g(x) & 0<x<L. \end{cases}$$

2 non-homog. initial conditions.

$t=1$

$y(x,1) \to$ displacement of particle at x at time 1





To solve Problem: use separation of variables. To get around 2 non-hom. conditions we split into 2 problems:

Problem A:

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = f(x) \\ y_t(x,0) = 0 \end{cases}$$

Problem B:

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0 \\ y_t(x,0) = g(x) \end{cases}$$

Find sols, y_A & y_B check:

$$y = y_A + y_B$$

solves original problem.