<u>Plan:</u> (5.2)

- Solving linear systems with distinct real eigenvalues
- Examples, elementary row operations

n×n

If $\underline{x}_1(t), \dots, \underline{x}_n(t)$ are lin. indep. sols of $\underline{x}' = \underline{P}(t)\underline{x}$

<u>then</u> any other sol'n of $\underline{x}' = \underline{P}(t)\underline{x}$ is of form

$$\underline{x} = c_1 \underline{x}_1(t) + \dots + c_n \underline{x}_n(t)$$

const., possibly complex.

$P(t) = \underline{A}$ const. matrix, real entries

<u>Ex:</u> $\underline{A} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}_{\text{no } t \text{ dependence.}}$, $\underline{x}' = \underline{A}\,\underline{x}$

Would need 2 lin. indep. sols.

unknown

<u>Remember:</u> $y'' + 2y' + 3y = 0$ tried $y = e^{rt}$, found char. eq'n, found $r$.

<u>Today:</u> hope that a sol'n to $\underline{x}' = \underline{A}\,\underline{x}$ looks like $\underline{x} = e^{\lambda t}\underline{v}$ ← unknown vector

unknown scalar

$$\underline{x}' = \underline{A}\,\underline{x} \Rightarrow \lambda e^{\lambda t}\underline{v} = \underline{A}(e^{\lambda t}\underline{v})$$

identity, n×n

$$e^{\lambda t} \neq 0 \underset{\text{scalar}}{\Longrightarrow} \lambda \underline{v} = \underline{A}\,\underline{v} \Rightarrow \underline{A}\,\underline{v} = \lambda \underline{I}\,\underline{v}$$

$\Rightarrow (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}} = \underline{\underline{0}}.$

If we can find $\lambda \in \mathbb{C}$ (or $\mathbb{R}$) so that there exists $\underline{\underline{v}} \neq 0$ satisfying $(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}} = \underline{\underline{0}}$ then

$$\underline{\underline{x}} = e^{\lambda t} \underline{\underline{v}}$$

would be a (non-trivial) solution of $\underline{\underline{x}}' = \underline{\underline{A}}\, \underline{\underline{x}}$.

Want: non-zero sol'n to $(\underset{n \times n}{\underline{\underline{A}}} - \lambda \underline{\underline{I}}) \underline{\underline{v}} = \underline{\underline{0}}$

This happens when

(1) $$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 \quad \leftarrow \text{characteristic eq'n of } \underline{\underline{A}}.$$

A $\lambda$ (real, complex or 0) which satisfies (1) is called an eigenvalue of $\underline{\underline{A}}$.

An eigenvector associated w/ an eigenvalue $\lambda$ is a non-zero vector $\underline{\underline{v}}$ so that

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}} = 0 \iff \underline{\underline{A}}\,\underline{\underline{v}} = \lambda \underline{\underline{v}}$$

Ex: $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\underline{\underline{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\underline{\underline{A}}\,\underline{\underline{v}} = 1 \cdot \underline{\underline{v}}$$

**Method** Given $\underline{x}' = \underline{\underline{A}}\,\underline{\underline{x}}$, $\underline{\underline{A}}$ const. matrix, $n\times n$.

1. Solve characteristic equation.
$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$
→ find eigenvalues $\lambda_1, \dots, \lambda_n$ ($n$ of them, possibly complex or repeated)

2. Find associated eigenvectors $\underline{\underline{v}}_1, \dots, \underline{\underline{v}}_n$

3. If process gives $n$ linearly eigenvectors then:
$$x_1(t) = e^{\lambda_1 t}\underline{\underline{v}}_1, \dots, x_n(t) = e^{\lambda_n t}\underline{\underline{v}}_n$$
are lin. indep. sol's of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$

4. Any sol'n is of form
$$\underline{x} = c_1 \underline{\underline{x}}_1(t) + \dots + c_n \underline{\underline{x}}_n(t)$$

**Fact**

If $\lambda_1, \dots, \lambda_n$ are all distinct (all different from each other) then step 3 works.

Ex: $\underline{\underline{A}} = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$ $\quad \underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$

$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. Find eigenvalues

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det \begin{bmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{bmatrix}$$

$$= (5-\lambda) \begin{vmatrix} -1-\lambda & -2 \\ -2 & -4-\lambda \end{vmatrix} + 0$$

$$-6 \begin{vmatrix} 2 & -1-\lambda \\ 4 & -2 \end{vmatrix}$$

$$= (5-\lambda)\Big( (-1-\lambda)(-4-\lambda) - 4 \Big) - 6\big(-4 - 4(-1-\lambda)\big)$$

$$= \; ---- =$$

$$= \lambda - \lambda^3$$

<u>So</u>:

$$\det(\underline{\underline{A}} - \lambda I) = 0 \iff \lambda - \lambda^3 = 0 \iff \underbrace{1, -1, 0}_{\substack{\text{distinct!} \\ \text{method works}}}$$

2. <u>Find eigenvectors</u>

$\lambda = 0$

$$(\underline{\underline{A}} - 0 \underline{\underline{I}}) \underline{\underline{v}} = \underline{\underline{0}}$$

want:

$$\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 5v_1 - 6v_3 = 0 & (1) \\ 2v_1 - v_2 - 2v_3 = 0 & (2) \\ \cancel{4v_1 - 2v_2 - 4v_3 = 0} & (3) \end{cases}$$

multiple of (2), no new info.

$(1) \Rightarrow v_1 = \frac{6}{5} v_3$

$(2) \Rightarrow v_2 = 2 \, \frac{6}{5} v_3 - 2v_3$

so:
$$\begin{bmatrix} \frac{6}{5} v_3 \\ \frac{12}{5} v_3 - 2v_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} - 2 \\ 1 \end{bmatrix} v_3$$

we have an eigenvector for any $v_3 \neq 0$
take $v_3 = 5$, an e.vector associated w/ $\lambda = 0$ is

$$\underline{v} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}, \quad \text{so} \quad c_1 e^{0t} \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} \text{ solves } \underline{x}' = \underline{A} \, \underline{x}.$$

$\underline{\underline{\lambda = 1}}$

$$(\underline{\underline{A}} - 1 \cdot \underline{\underline{I}})\, \underline{v} = 0$$

$$\begin{bmatrix} 5-1 & 0 & -6 \\ 2 & -1-1 & -2 \\ 4 & -2 & -4-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Elementary row operations:
1. interchange rows
2. multiply row by non-zero number
3. add multiple of a row to another.

1st goal: have 1 at top left, 0's under it.

$$\left[\begin{array}{ccc|c} 4 & 0 & -6 & 0 \\ 2 & -2 & -2 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

$(2) \leftrightarrow (1)$

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 4 & 0 & -6 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

$\frac{1}{2}$ ① → ①

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 4 & 0 & -6 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

② − 4·① → ②

③ − 4·① → ③

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

③ − $\frac{1}{2}$② → ③

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Equiv. system:

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_3 = 2v_2$

$v_1 = v_2 + 2v_2 \implies v_1 = 3v_2$

$V = \begin{bmatrix} 3v_2 \\ v_2 \\ 2v_2 \end{bmatrix}$ is an eigenvector. Take $v_2 = 1$

$\underset{=}{x}(t)$ $e^t \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ is a sol'n.

Ex: do case $\lambda = -1$, find eigenv. $\underset{=}{v}_3$

General sol'n of $\underset{=}{x}' = \underset{=}{A}\,\underset{=}{x}$

$$\underset{=}{x} = c_1 \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} + c_2 e^t \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{-t}\, \underset{=}{v}_3$$

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