Vibrating String Eq'n: string of length L.

$$\begin{cases} y_{tt} = a^2 y_{xx} & 0 < x < L, \quad t > 0 \\ y(0,t) = y(L,t) = 0 & \leftarrow \text{fix endpts} \\ y(x,0) = f(x) & (\text{non-homog. initial cond}) \\ y_t(x,0) = 0 & (\text{homog. initial cond.}) \end{cases}$$

Call sol'n y_A

Found: $$y_A(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \quad (*)$$

w/ $A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx$ ← F. sine series coef.

Trig. identity:

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

Combine w/ $(*)$:

$$y_A(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \left(\sin\left(\frac{n\pi}{L}(x+at)\right) + \sin\left(\frac{n\pi}{L}(x-at)\right)\right)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}(x+at)\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}(x-at)\right)\right)$$

Note: $f_{odd}(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$ (bec. A_n coef. of F. sine series)

So: $y_A(x,t) = \frac{1}{2}\left(f_{odd}(x+at) + f_{odd}(x-at)\right)$

wave traveling left w/ speed a

wave traveling right w/ speed a

D'Alembert formula

w/ non-homog. initial velocity

$$\begin{cases} y_{tt} = a^2 y_{xx} & 0 < x < L, \quad t > 0 \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0 \\ y_t(x,0) = g(x) \end{cases}$$

w/ separation of variables

Sol'n: $y_B(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{an\pi}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$

$$B_n = \frac{2}{na\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

not same as F. sine series coef. of $g(x)$.

Ex:

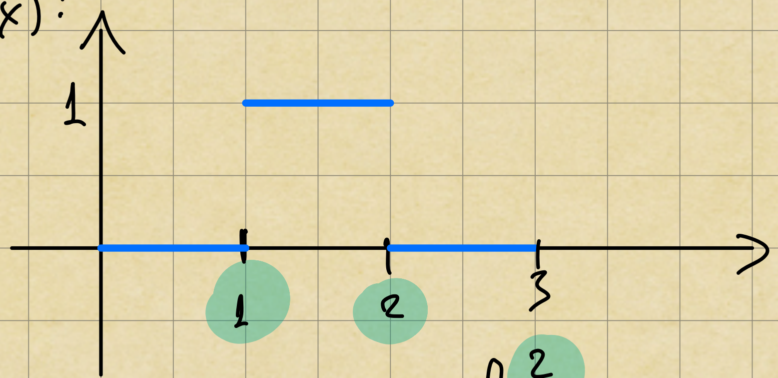
$$y_{tt} = 4y_{xx} \qquad 0 < x < 3, \; t > 0$$

$$y(0,t) = y(3,t) = 0$$

$$y(x,0) = 0$$

$$y_t(x,0) = g(x)$$

$g(x)$:



$$B_n = \frac{2}{n \cdot \underbrace{2}_{= a} \pi} \int_1^2 \sin\left(\frac{n\pi}{3} x\right) dx$$

$$= \frac{1}{\pi n} \cdot \frac{3}{\pi n} \left(\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2\pi n}{3}\right) \right)$$

So: $$y_B(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n}{3} t\right) \sin\left(\frac{\pi n}{3} x\right)$$

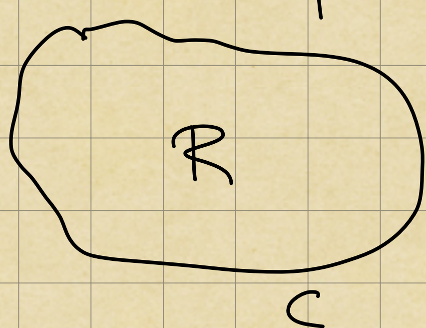
w/ B_n we found. //

Rmk: $y = y_A + y_B$ solves

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = f(x) \\ y_t(x,0) = g(x) \end{cases}$$

9.7 Laplace Eq'n.

Setting: thin 2-dim'l plate (lamina). Occupies domain R in the plane



C: boundary of domain (nice)

Faces of lamina are insulated (heat can flow in and out only from the boundary)

Model temperature $u(\underbrace{x,y}_{(x,y) \text{ coord. of a point}}, \underbrace{t}_{\text{time.}})$

u satisfies the <u>heat</u> equation in 2 dims:

$$\boxed{\frac{\partial u}{\partial t} = k \nabla^2 u} \qquad (*)$$

where

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Laplace operator / Laplacian

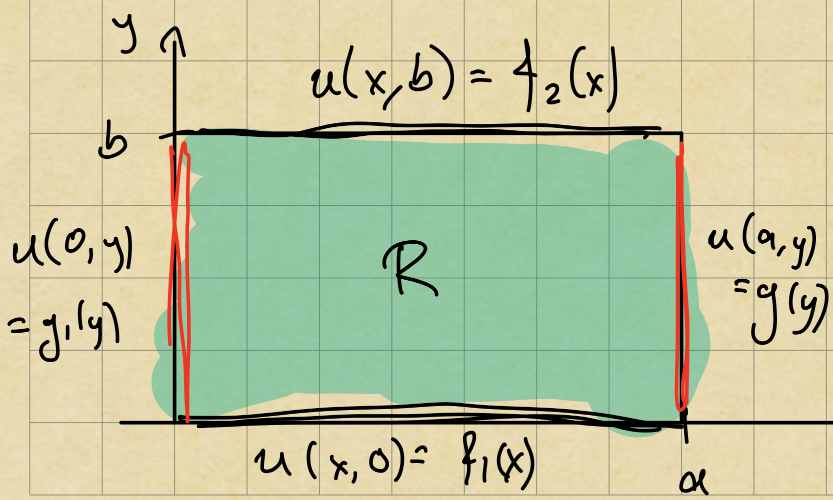
Another notation: Δu

k: thermal diffusivity.

We will look at the case where u const. in time.

$(*) \Longrightarrow \nabla^2 u = 0$ Laplace eq'n. (potential eq'n)

Domain will be a rectangle.



In R: $\nabla^2 u = 0$

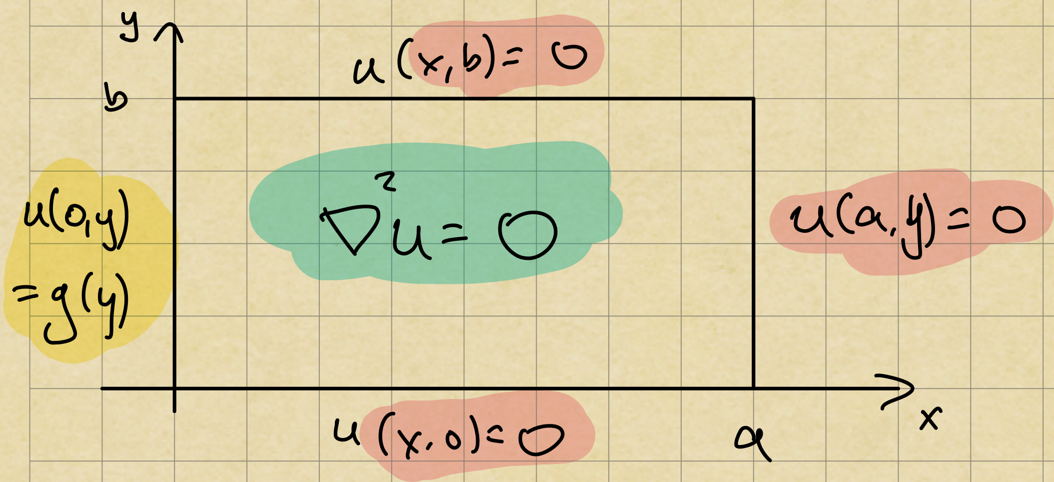
To determine a sol'n: prescribe boundary conditions.

Terminology: A problem of the form

$$\nabla^2 u = 0 \quad \text{in } R$$

w/ values of u prescribed in the boundary of R is called a Dirichlet problem. If boundary & boundary values are nice, it has unique sol'n.

Start w/ Dirichlet problem with only one non-homog. condition.



Seek sol'n in form $\sum c_n u_n(x,y)$.

Assume: $u_n(x,y) = X_n(x)\, Y_n(y)$. $\triangle$

Want:

$$\nabla^2 u_n = 0 \quad ①$$

$$u_n(x,0) = u_n(x,b) = u_n(a,y) = 0$$

Take $\triangle$, plug into $\square$: separate variables (x on one side, y on the other).
Find dif. eq's satisfied by X_n, Y_n.

$$\nabla^2 u_n = 0 \Rightarrow X_n'' Y_n + X_n Y_n'' = 0$$

$$\Rightarrow \quad \underbrace{\frac{X_n''}{X_n}}_{\text{depends only on } x} = - \underbrace{\frac{Y_n''}{Y_n}}_{\text{depends only on } y} = -\lambda$$

both const.

$$\begin{cases} X_n'' = -\lambda X_n \\ Y_n'' = \lambda Y_n \end{cases}$$