Final: in 1 week.

No OH today. OH tomorrow 3-5.

Sturm-Liouville Problems

$$\begin{cases} x'' + 4x = f(x) \\ x(0) = x(L) = 0 \end{cases}$$

Found F. sine series for $f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

known: $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

Expanded x using $y_n = \sin\left(\frac{n\pi}{L}x\right)$ as building blocks:

$$x = \sum_{n=1}^{\infty} A_n y_n(x)$$

A_n: tbd

Notice: $y_n(0) = y_n(L) = 0$ (1)

$\Rightarrow x(0) = x(L) = 0$

good property: satisfy endpt cond.

$$x'' = \sum_{n=1}^{\infty} A_n y_n''(x)$$

$$= \sum_{n=1}^{\infty} A_n\left(-\left(\frac{n\pi}{L}\right)^2\right) y_n$$

good property

[bec $y_n'' = -\left(\frac{n\pi}{L}\right)^2 y_n$]

$$\Rightarrow \quad x'' + 4x = \sum_{n=1}^{\infty} A_n \left(-\left(\frac{n\pi}{L}\right)^2 + 4 \right) y_n$$

$$= \sum_{n=1}^{\infty} b_n y_n$$

$$\Rightarrow \quad A_n = \frac{b_n}{\left(-\left(\frac{n\pi}{L}\right)^2 + 4 \right)}$$

Sturm-Liouville problems: generalize this, create building blocks w/ good eigenfunction properties and good endpt conditions.

<u>Sturm-Liouville problem</u>

$$\begin{cases} \frac{d}{dx}\left[p(x) \frac{dy}{dx} \right] - q(x) y + \lambda r(x) y = 0 \quad \text{on } a < x < b \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

Neither α_1 and α_2 both 0, nor β_1 and β_2 both 0.

λ: eigenvalue, to be determined.

y: eigenfunction, if it is non-trivial.

In this class: only $y'' + \lambda y = 0$

$p \equiv 1, \ q \equiv 0, \ r \equiv 1$

Ex 1: $y'' + \lambda y = 0$

$y(0) = y(L) = 0$

Seen: non-trivial sols exist exactly when

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $y = \sin\left(\frac{n\pi}{L}x\right)$

is an eigenfunction.

Ex 2: $y'' + \lambda y = 0$

$y(0) = 0$

$y'(L) = 0.$

$$\begin{cases} p \equiv 1, \ q \equiv 0, \ r \equiv 1 \\ a = 0, \quad b = L \\ \alpha_1 = 1, \ \alpha_2 = 0 \\ \beta_1 = 0, \ \beta_2 = 1 \end{cases}$$

$\lambda = 0$: $\Rightarrow y'' = 0 \Rightarrow y = Ax + B$

$y(0) = 0 \Rightarrow B = 0$

$y'(L) = 0 \Rightarrow A = 0$

So: if $\lambda = 0$ no non-trivial sol's

and $\lambda = 0$ is not an eigenvalue.

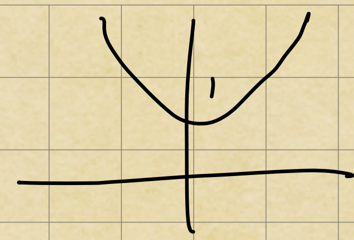
$\lambda = -a^2$ $\quad y'' - a^2 y = 0$

$y = e^{\pm ax}$

$y(x) = A\cosh(ax) + B\sinh(ax)$

$y(0)=0 \Rightarrow A=0$

$y(x) = B\sinh(ax)$



$\rightarrow y'(x) = aB\cosh(ax)$

Want: $y'(L)=0 \Rightarrow aB\cosh(aL)=0$

$\Rightarrow B=0.$

So $\lambda = -a^2$ is not an eigenvalue
(no non-trivial sols)

$\lambda = a^2$ $\quad y'' + a^2 y = 0$

$$y = A\cos(ax) + B\sin(ax)$$

$y(0)=0 \Rightarrow A=0$

$$y'(L)=0 \Rightarrow \underbrace{aB}_{\text{want } \neq 0}\overbrace{\cos(aL)}^{\text{want } =0} = 0$$

$\Rightarrow aL = (2n+1)\frac{\pi}{2}$
$n \geq 0$ integer.

So: $a = \frac{2n+1}{2L}\pi$ and

$\lambda = \left(\frac{2n+1}{2L}\pi\right)^2$ is an eigenvalue for all $n \geq 0$ integer

w/ cor. eigenfct $y(x) = \sin\left(\frac{2n+1}{2L}\pi x\right)$ //

Rmk: if y_n is an eigenfct then By_n is an an eigenfct for any const. B.

Thm:

$$\begin{cases} \frac{d}{dx}\left[p(x) \frac{dy}{dx} \right] - q(x) y + \lambda r(x) y = 0 \quad \text{on } a < x < b \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

If p, q, r nice
$p > 0$, $r > 0$

Then: eigenvalues form an increasing sequence

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

and $\lim_{n \to \infty} \lambda_n = \infty$.

There is a single eigenfunction cor. to each eigenvalue, up to multiplication by a constant.

If $q \geq 0$ on $[a,b]$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$
$\Rightarrow \lambda_j \geq 0$

Ex:

$$y'' + \lambda y = 0 \qquad y(0) = y'(L) = 0$$

$$\begin{cases} p \equiv 1, q \equiv 0, r \equiv 1 \\ a = 0, \quad b = L \\ \alpha_1 = 1, \alpha_2 = 0 \\ \beta_1 = 0, \beta_2 = 1 \end{cases}$$

$\Rightarrow$ by theorem $\lambda_j \geq 0$, as we found.

$$\underbrace{\lambda_n = \left(\frac{2n+1}{2L}\pi\right)^2}_{\text{increasing seq.}}, \quad \lambda_n \xrightarrow{n\to\infty} \infty$$

Rmk: Eigenvalues depend on endpt conditions, even for the same ODE: Ex 1 & 2.