Last time:

$$\begin{cases} \dfrac{d}{dx}\left(p(x)\dfrac{dy}{dx}\right) - q(x)y + \lambda r(x) y = 0 & a < x < b \\ \underbrace{\alpha_1 y(a) - \alpha_2 y'(a)}, \quad \underbrace{\beta_1 y(b) + \beta_2 y'(b)} = 0 \end{cases}$$

2nd order ODE

endpt conditions

$\boxed{\text{Thm 1}}$

If p, q, r nice, $p > 0$, $r > 0$ (works for $y'' + \lambda y = 0$) then e-values increasing

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$$

$$\lim_{n\to\infty} \lambda_n = \infty$$

If $\underline{q \geq 0}$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ all $\geq 0 \Rightarrow \lambda_j \geq 0$

works for $y'' + \lambda y = 0$

regular S-L problem.

Ex: $\quad y'' + \lambda y = 0$

$\quad hy(0) - y'(0) = 0 \qquad y(L) = 0$

$$h > 0$$
$$L > 0$$

Look for e-values and e-fcts.

Note: $p \equiv 1$, $r \equiv 1$, $q \equiv 0$

$\alpha_1 = h \qquad \beta_1 = 1$

$\alpha_2 = 1 \qquad \beta_2 = 0$

So $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0 \Rightarrow$ regular S-L.

$\lambda_j \geq 0$.

Find eigenvalues.

$\underline{\lambda = 0}$ $\quad y'' = 0$

$hy(0) - y'(0) = 0, \quad y(L) = 0$

Are there non-trivial sols?

$y'' = 0 \Rightarrow \quad y = Ax + B$

$h\,y(0) - y'(0) = 0 \Rightarrow hB - A = 0$

$y(L) = AL + B = 0 \Rightarrow B = -AL$

So $\quad -hAL - A = 0 \Rightarrow A(\underbrace{-hL - 1}_{<0}) = 0$

$\Rightarrow A = 0.$

So $B = 0.$

So $\lambda_0 = 0$ <u>not</u> an eigenvalue.

$\rightarrow \lambda = a^2 > 0. \quad y'' + a^2 y = 0$

$y = A\cos(ax) + B\sin(ax)$

$\rightarrow h\underbrace{A}_{y(0)} - \underbrace{aB}_{y'(0)} = 0 \Rightarrow A = \dfrac{aB}{h}$

$\rightarrow \quad y(L) = 0 \Rightarrow A\cos(aL) + B\sin(aL) = 0.$

$$\Rightarrow B\left(\frac{a}{h}\cos(aL) + \sin(aL)\right) = 0$$

<u>Want</u>: find a so that $B \neq 0$ (non-trivial sol's exist)

<u>Want:</u>

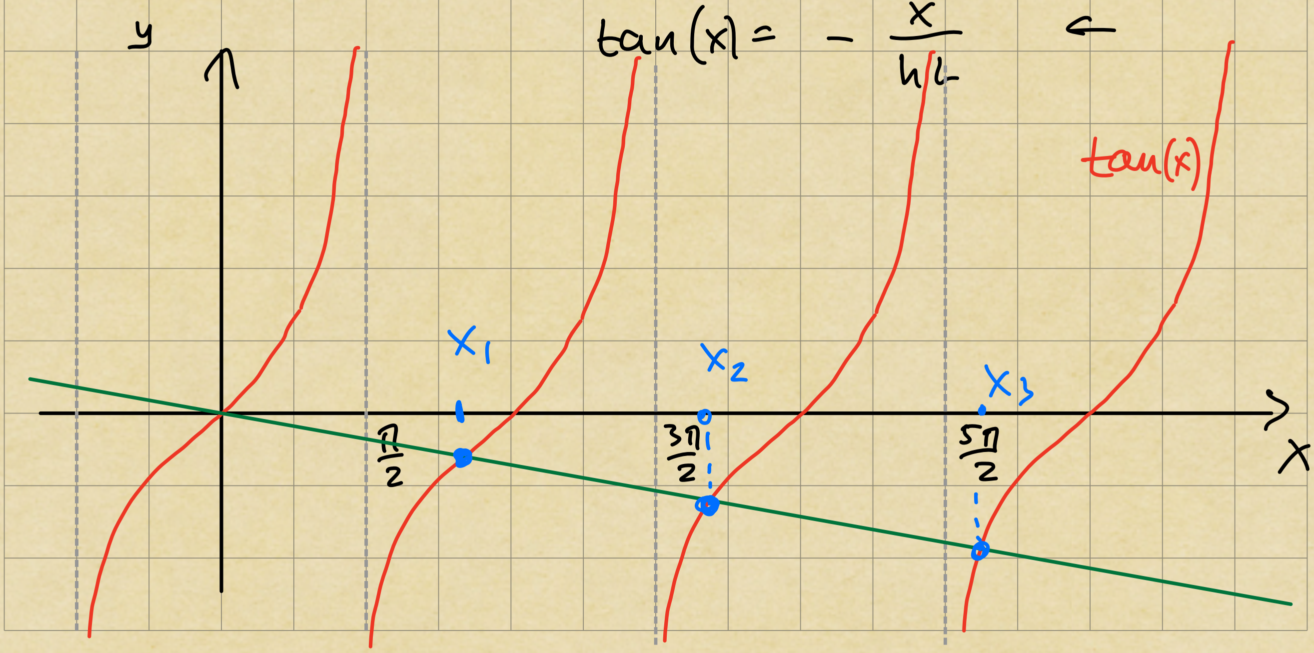
$$\frac{a}{h}\cos(aL) + \sin(aL) = 0$$

$$\Rightarrow \quad \frac{a}{h}\cos(aL) = -\sin(aL)$$

$$\Rightarrow \quad \tan(aL) = -\frac{aL}{hL} \qquad (*)$$

<u>Note:</u> a solves $(*)$ exactly when $a = \frac{x}{L}$, where x solves

$$\tan(x) = -\frac{x}{h/L} \quad \leftarrow$$



<u>So:</u> let x_n the nth positive sol'n of $\tan x = -\frac{x}{hL}$

then eigenv. are $\lambda_n = a_n^2 = \left(\frac{x_n}{L}\right)^2$

Eigenfcts: $\frac{a_n}{h}\cos(a_n x) + \sin(a_n x)$

$\frac{x_n}{Lh}\cos\left(\frac{x_n}{L}x\right) + \sin\left(\frac{x_n}{L}x\right)$

took $B=1$ (any const. multiple of an e-fct is an e-fct. //

E-fcts as building blocks for expressing fcts.

<u>Fact 1:</u> E-fct cor. to different evalues are orthogonal.

w/ assumptions of Thm 1:

$$\int_a^b y_i(x)\, y_j(x)\, r(x)\, dx = 0 \qquad i \neq j$$

<u>Ex:</u> $y'' + \lambda y = 0$ $\qquad r(x) \equiv 1$

$y(0) = y(\pi) = 0$

E-values: $\lambda = n^2$, $y_n = \sin(nx)$.

Check: $$\int_0^{\pi} \sin(nx)\sin(mx)\,dx = 0 \qquad n \neq m.$$

Fact 2 Eigenfcts of regular S-L problems can be used as building blocks to represent functions.

$$f(x) = \sum_{m=0}^{\infty} c_m\, y_m(x)$$

where $$c_m = \frac{\int_a^b f(x)\, y_m(x)\, r(x)\,dx}{\int_a^b (y_m(x))^2\, r(x)\,dx}$$

If f piecewise smooth, sum converges to

$\to f(x)$ (f cont. at x)

$\to \frac{1}{2}\left(f(x^-)+f(x^+)\right)$ (f discont. at x)

Ex: $y'' + \lambda y = 0$

$y(0) = y'(1) = 0$

Saw:

$$\lambda_n = \left(\frac{2n-1}{2}\pi\right)^2, \quad y_n = \sin\left(\frac{2n-1}{2}\pi x\right)$$

$n \geq 1$ integer.

Represent $f(x) = 1$ on $[0,1]$.

$r(x) \equiv 1$

$$c_m = \frac{\int_0^1 1 \sin\left(\frac{2m-1}{2}\pi x\right)dx}{\int_0^1 \sin^2\left(\frac{2m-1}{2}\pi x\right)dx}$$

$$\int_0^1 1 \sin\left(\frac{2m-1}{2}\pi x\right)dx = -\frac{2}{\pi(2m-1)} \cos\left(\frac{2m-1}{2}\pi x\right)\Big|_0^1$$

$$= -\frac{2}{\pi(2m-1)}\left(\underbrace{\cos\left((2m-1)\frac{\pi}{2}\right)}_{0} - \cos(0)\right)$$

$$= \frac{2}{\pi(2m-1)}.$$

$$\int_0^1 \sin^2\left(\frac{2m-1}{2}\pi x\right)dx \overset{\text{double angle}}{=} \int_0^1 \frac{1-\cos((2m-1)\pi x)}{2}dx$$

$$= \dots = \frac{1}{2}$$

So: $$f(x) = \sum_{n=1}^{\infty} \frac{\frac{2}{2n-1}}{\frac{1}{2}} \sin\left(\frac{2n-1}{2}\pi x\right)$$