2009 Q5, 18, 10

9.4 #4

2009

5: $\varphi' - \int_0^t (t-\xi)^2 \varphi(\xi)\, d\xi = \delta(t-3)$, $\varphi(0) = 1$

$\int_0^t \varphi(\tau)(t-\tau)^2 d\tau = \varphi(t) * t^2$

$\Phi(s) = ?$

$s\Phi(s) - 1 - \Phi(s) \cdot \mathcal{L}\{t^2\} = e^{-3s}$

$s\Phi(s) - 1 - \Phi(s)\frac{2}{s^3} = e^{-3s}$

$\Phi(s)\left(s - \frac{2}{s^3}\right) = e^{-3s} + 1$

$\Phi(s) = \frac{1}{s - \frac{2}{s^3}} e^{-3s} + \frac{1}{s - \frac{2}{s^3}}$ //

$\mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a) f(t-a)$

10:

$$x' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} x$$

$\lambda = 1$ mult. 1 $\rightarrow$ $\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 3$ mult. 2. $\rightarrow$ $\eta = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$x(t) = \underbrace{t e^{3t} \xi + e^{3t} \eta}_{\lambda = 3 \text{ defect } 1.}$ $\qquad \xi = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\boxed{\eta}$ $\leftarrow$ generalized eigenvector

$\downarrow$

$\boxed{\xi}$ — $\boxed{\zeta}$

$\uparrow$ $\nearrow$

eigenvectors

$(A - 3I)^2 \eta = 0$

$(A - 3I)\eta \neq 0$

If η known

then

$\xi = (A - 3I)\eta$

$$\underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{\xi} = \underbrace{\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{A-3I} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

$-2\eta_1 + \eta_2 = 1$

$\eta_3 = 2$

Ans: B: only one option doesn't satisfy

18

$$u_{xx} = u_t$$

$u(x,0) = f(x) \to$ some fct.

$u(0,t) = 20 \quad , u(5,t) = 70$

Steady sol'n: $\partial_t u = 0$

$$\begin{cases} {}^s u_{xx} = 0 & \leftarrow \text{ODE} \\ {}^s u(0) = 20 \\ {}^s u(5) = 70 \end{cases}$$

$${}^s u''(x) = 0 \quad \Rightarrow \quad {}^s u(x) = Ax + B$$

${}^s u(0) = 20 \Rightarrow B = 20$

${}^s u(5) = 70 \Rightarrow 5A + 20 = 70$

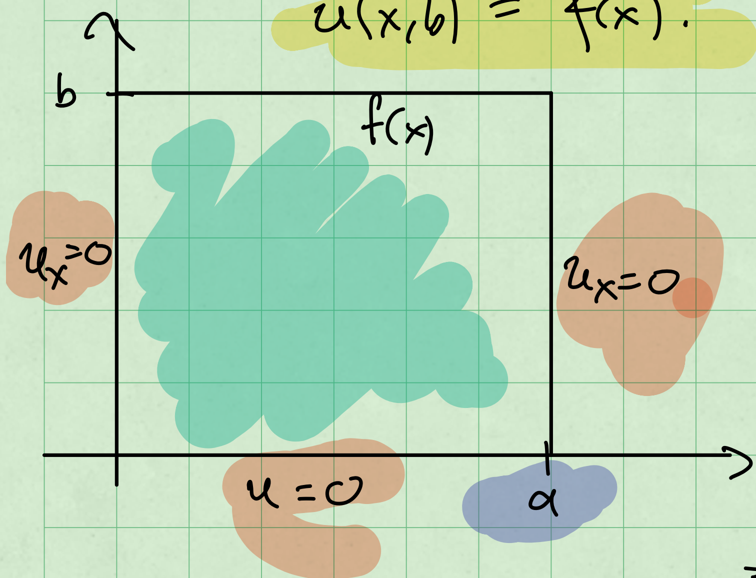
$\Rightarrow A = 10$

$${}^s u = 10x + 20$$

Ans: A.

§9.7 #4.

$$u_{xx} + u_{yy} = 0$$
$$u_x(0,y) = u_x(a,y) = u(x,0) = 0$$
$$u(x,b) = f(x).$$



$$u = \sum_{n=0}^{\infty} c_n X_n(x) Y_n(y)$$
$$u_{xx} + u_{yy} = 0$$
$$\Rightarrow X_n'' Y_n + X_n Y_n'' = 0$$

$$\Rightarrow \frac{Y_n''}{Y_n} = -\frac{X_n''}{X_n} = \lambda$$

→ 2 odes

$$\begin{cases} Y_n'' - \lambda Y_n = 0 \\ X_n'' + \lambda X_n = 0 \end{cases}$$

Look for endpt cond:

$u_x(0,y) = 0 \Rightarrow \quad X_n'(0) Y_n(y) = 0$

$\Rightarrow X_n'(0) = 0$

$u_x(a,y) = 0 \Rightarrow \dots \Rightarrow X_n'(a) = 0$

$u(x,0) = 0 \Rightarrow Y_n(0) = 0$

$$\begin{cases} X_n'' + \lambda X_n = 0 \\ X_n'(0) = X_n'(a) = 0 \end{cases} \quad 1$$

$$\begin{cases} Y_n'' - \lambda Y_n = 0 \\ Y_n(0) = 0 \end{cases}$$

2

1 $\rightarrow$ $\lambda = 0$

$$X_n'' = 0 \Rightarrow X_n = Ax + B$$

$$X_n'(0) = X_n'(a) = 0 \Rightarrow A = 0.$$

Can take $B = 1$ and $X_0 = 1$ e-fctn.

$\rightarrow$ $\lambda = \alpha^2$

$$X_n = \tilde{A}\cos(\alpha x) + \tilde{B}\sin(\alpha x)$$

$X_n'(0) = 0 \Rightarrow \tilde{B} = 0$

$X_n'(a) = 0 \Rightarrow -\alpha\tilde{A}\sin(\alpha a) = 0$

$\alpha a = n\pi$

$\Rightarrow \alpha = \frac{n\pi}{a}$

$\alpha = \frac{n\pi}{a}$, $\tilde{B} = 0$, $\tilde{A}$ anything

$$X_n(x) = \cos\left(\frac{n\pi}{a}x\right), \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2$$

Now: (2)

$$Y_0'' = 0 \Rightarrow Y_0 = Ay + B$$

$$Y(0) = 0 \qquad Y(0) = 0 \Rightarrow B = 0$$

A anything, take $A=1$

$$Y_0 = y$$

$$Y_n'' = \left(\frac{\pi n}{a}\right)^2 Y_n \Rightarrow$$

$$Y_n = A\cosh\left(\frac{n\pi}{a}y\right) + B\sinh\left(\frac{n\pi}{a}y\right)$$

$$Y_n(0) = 0 \Rightarrow A = 0$$

can take

$$Y_n = \sinh\left(\frac{n\pi}{a}y\right)$$

Therefore:

$$u(x,y) = c_0 \cdot 1 \cdot y + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}y\right)\cos\left(\frac{n\pi}{a}x\right)$$

have to be determined from non-hom. bd cond.

Now want: $u(x,b) = f(x)$

known

$$f(x) = \underbrace{b\,c_0}_{} + \sum_{n=1}^{\infty} \underbrace{c_n \sinh\left(\frac{n\pi}{a} b\right)}_{\text{const.}} \cos\left(\frac{n\pi}{a} x\right)$$

$$b\,c_0 = \frac{a_0}{2}, \quad \text{where} \quad a_0 = \frac{2}{a}\int_0^a f(x)\,dx$$

$$c_n \sinh\left(\frac{n\pi}{a} b\right) = a_n, \quad \text{where} \quad a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi}{a} x\right) dx$$

$a_0, a_n \to$ Fourier cosine series coef. of $f(x)$.