Plan: Finish phase plane portraits for systems w/ real e-values (5.3 up to p. 303)
Discuss systems with complex e-values (5.2 p 289 and on) & corresponding phase plane portraits (5.3 p. 312 and on)

Last time: phase plane portraits, distinct real e-values

Cases: opposite signs (saddle)
2 negative e-values (nodal sink)
2 positive e-values (nodal source)

Today: finish distinct real e-values

Ex: one 0, one negative e-value.

$$\begin{cases} x' = -6x + 6y \\ y' = 9x + 9y \end{cases}$$

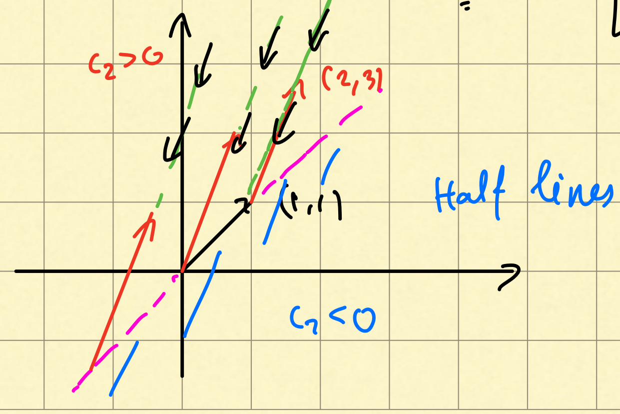
check: e-values: 0, -3

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$c_1 = 1, \; c_2 = 1$

$$\underline{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

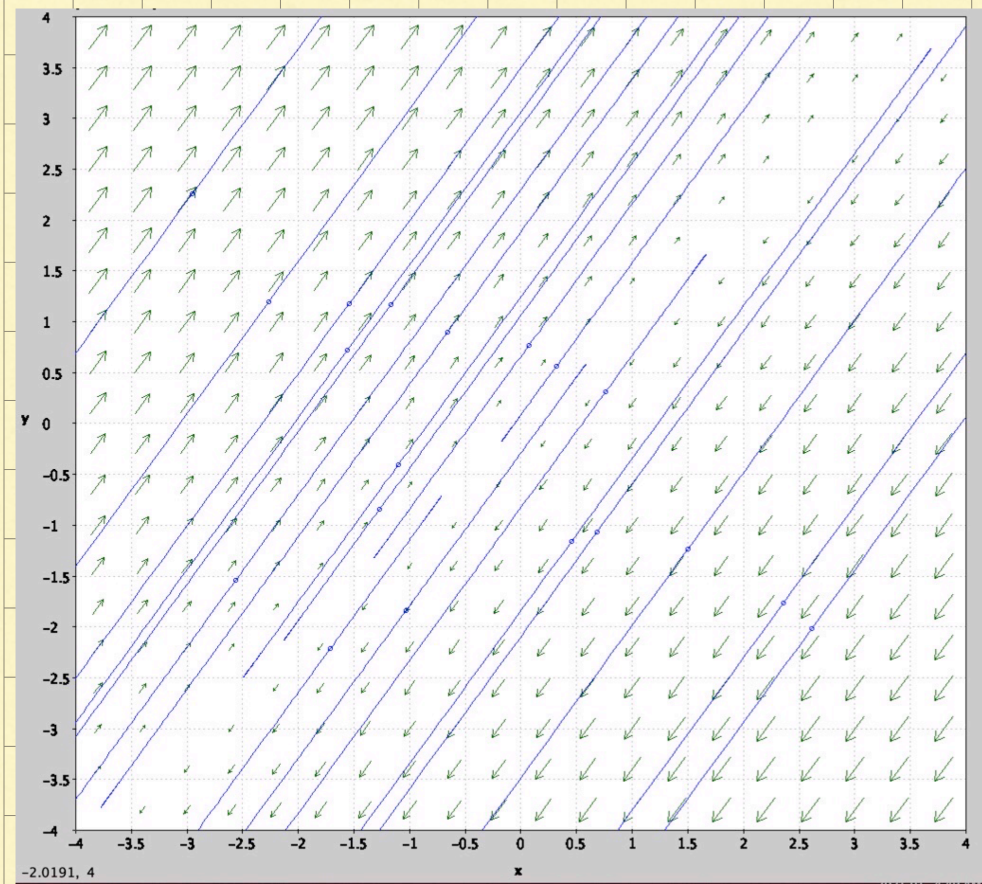
$$\underline{x}'(t) = -3 c_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



Ex: one 0, one positive e-value:

$$\begin{cases} x' = 6x - 6y \\ y' = -9x - 9y \end{cases}$$

same picture w/ reversed green arrows.



5.3 up to p. 303

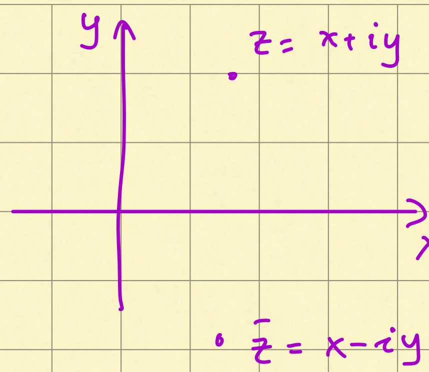
Back to 5.2, p. 289 and on

Complex #: $z = a + ib$, $a, b \in \mathbb{R}$

$i^2 = -1$

$Re(z) = a$, $Im(z) = \underbrace{b}_{\text{the imaginary part of } z \text{ is real!}}$

Conjugate: $\bar{z} = a - ib$



$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}, \quad \underline{\underline{A}} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} -3-\lambda & 4 \\ -4 & -3-\lambda \end{vmatrix} = (\lambda+3)^2 + 16 = 0$$

$$\Rightarrow (\lambda+3)^2 = -16$$

$$\Rightarrow \lambda + 3 = \pm\sqrt{-16}$$

$$\Rightarrow \underbrace{\lambda = -3 \pm 4i}$$

the two roots are conjugate to each other.

Looking for an e-vector assoc. to $\lambda = -3+4i$:

$$\begin{bmatrix} -3-(-3+4i) & 4 \\ -4 & -3-(-3+4i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4i\,v_1 + 4v_2 = 0 \quad \Rightarrow \quad v_1 = \frac{1}{i} v_2$$

$$-4v_1 - 4i\,v_2 = 0 \quad \Rightarrow \quad v_1 = -i\,v_2$$

Find reciprocal of cplx #: $\dfrac{1}{a+bi} = \dfrac{a-bi}{(a+bi)(a-bi)}$

$$= \frac{a-bi}{a^2+b^2}$$

so: $\dfrac{1}{i} = \dfrac{-i}{\underbrace{i(-i)}_{=1}} = -i$

So e-vector is $\begin{bmatrix} -i \\ 1 \end{bmatrix} v_2$

So one sol'n of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ is given by

$$\underline{x}(t) = c_1 e^{(-3+4i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Need a second sol'n.

2 appoaches:

1. Find eigenvector $\underline{\underline{v_2}}$ assoc. to $\lambda = -3-4i$

gen. sol'n:

$$\underline{x} = c_1 e^{(-3+4i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{(-3-4i)t} \underline{\underline{V_2}}$$

actually this is $\overline{\begin{bmatrix} -i \\ 1 \end{bmatrix}} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

go with this

2. Observation: if $\underline{\underline{A}}$ real entries and $\underline{x}(t)$ solves $\underline{x}'(t) = \underline{\underline{A}}\,\underline{x}(t)$ then

$$\left.\begin{aligned} \left(\mathrm{Re}\,\underline{x}(t)\right)' &= \underline{\underline{A}}\left(\mathrm{Re}\,\underline{x}(t)\right) \\ \left(\mathrm{Im}\,\underline{x}(t)\right)' &= \underline{\underline{A}}\left(\mathrm{Im}\,\underline{x}(t)\right) \end{aligned}\right\} \text{taking real \& imaginary parts entry-wise}$$

So from one sol'n we can produce two.

Take real & imaginary pt of

$$\underline{x}(t) = c_1 e^{(-3+4i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Euler's formula: $e^{a+bi} = e^{a}\left(\cos(b) + i\sin(b)\right)$

$$\underline{x}_1 = e^{(-3+4i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{-3t}\left(\cos(4t) + i\sin(4t)\right) \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= \underbrace{e^{-3t}}_{\text{real}} \begin{bmatrix} \sin(4t) - i\cos(4t) \\ \cos(4t) + i\sin(4t) \end{bmatrix}$$

$$\underline{y}_1 = \operatorname{Re} \underline{x}_1 = e^{-3t} \begin{bmatrix} \sin(4t) \\ \cos(4t) \end{bmatrix}, \quad \underline{y}_2 = \operatorname{Im} \underline{x}_1 = e^{-3t} \begin{bmatrix} -\cos(4t) \\ \sin(4t) \end{bmatrix}$$

So: $\underline{y}_1, \underline{y}_2$ are sol's. Are they lin. indep?

$$W(\underline{y}_1, \underline{y}_2) = \det \begin{bmatrix} e^{-3t}\sin(4t) & -e^{-3t}\cos(4t) \\ e^{-3t}\cos(4t) & e^{-3t}\sin(4t) \end{bmatrix}$$

$$= \ldots = e^{-6t} \neq 0 \quad \text{lin. indep.}$$

Gen. sol':

$$\underline{x} = c_1 e^{-3t} \begin{bmatrix} \sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -\cos(4t) \\ \sin(4t) \end{bmatrix}$$

<u>Summary:</u> (for cplx e-values)

→ Find e-values

→ Find <u>one</u> e-vector cor. to one

→ Find cplx valued sol'n $e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

→ Take real & imaginary parts to find 2 lin. indep. sols.