Plan:

→ Phase plane portraits of systems w/ complex eigenvalues (5.3 p. 312 and on)

→ Solving systems w/ real repeated eigenvalues (5.5)

Tuesday: 1 problem on Gradescope (+ online HW)

OH Tuesday 3-5

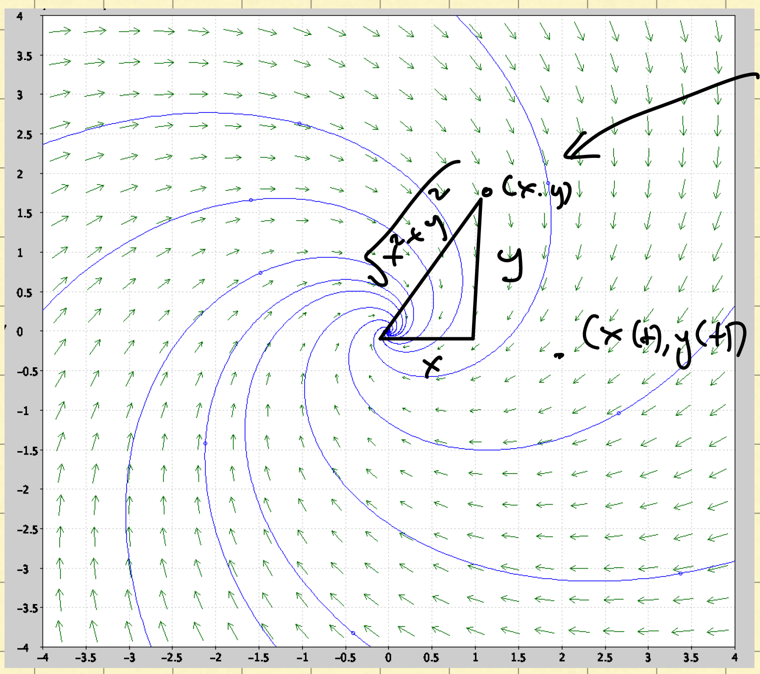
Phase plane portraits for systems w/ cplx e-values.

Picture depends on real pt of sols.

Ex: $x' = -3x + 4y$ $\qquad \lambda = \underbrace{-3 \pm 4i}_{\text{negative real pt.}}$

$y' = -4x - 3y$

Sol'n: $\underline{x}(t) = a e^{-3t} \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix} + b e^{-3t} \begin{bmatrix} \sin(4t) \\ -\cos(4t) \end{bmatrix}$



spiral sink

$$x(t) = a e^{-3t}\cos(4t) + b e^{-3t}\sin(4t) \quad ①$$
$$y(t) = a e^{-3t}\sin(4t) - b e^{-3t}\cos(4t) \quad ②$$

special case:

$a=1,\ b=0.$

$$x(t) = e^{-3t}\cos(4t), \quad y(t) = e^{-3t}\sin(4t)$$

"circle w/ variable radius"

Can also look at distance of $(x(t), y(t))$ from the origin, $(x^2(t) + y^2(t))^{1/2}$

①, ② $\Rightarrow$

$$x^2(t) = a^2 e^{-6t}\cos^2(4t) + 2ab e^{-6t}\cos(4t)\sin(4t) + b^2 e^{-6t}\sin^2(4t)$$
$$y^2(t) = a^2 e^{-6t}\sin^2(4t) - 2ab e^{-6t}\cos(4t)\sin(4t) + b^2 e^{-6t}\cos^2(4t)$$

$\cos^2 + \sin^2 = 1$

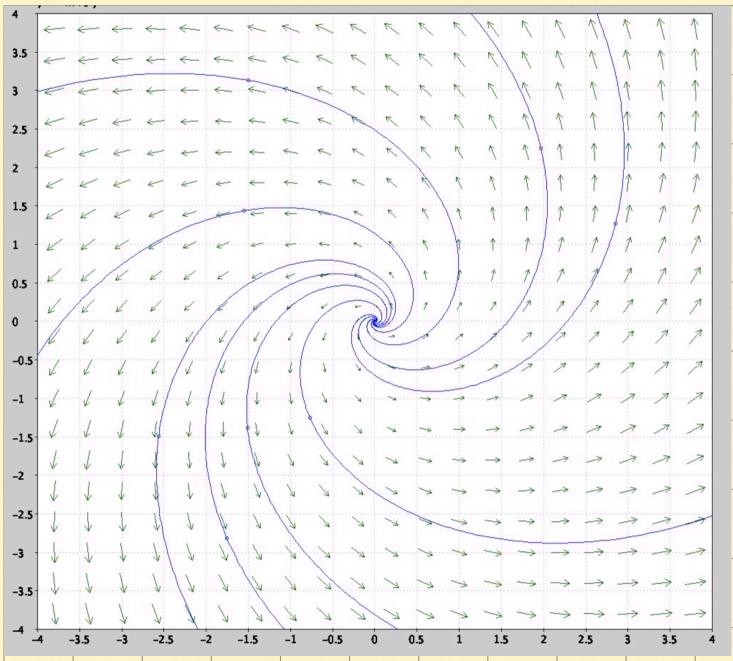
$$x^2(t) + y^2(t) = (a^2+b^2)e^{-6t} \xrightarrow[t\to\infty]{} 0$$

Ex: positive real pt:

$x' = 3x - 4y$

$y' = 4x + 3y$

$\lambda = 3 \pm 4i$



spiral source.

[In response to question in class: example of system whose phase plane portrait has trajectories spiraling towards the origin counterclockwise:

$$\begin{cases} x' = -4x - 3y \\ y' = 3x - 4y \end{cases}$$

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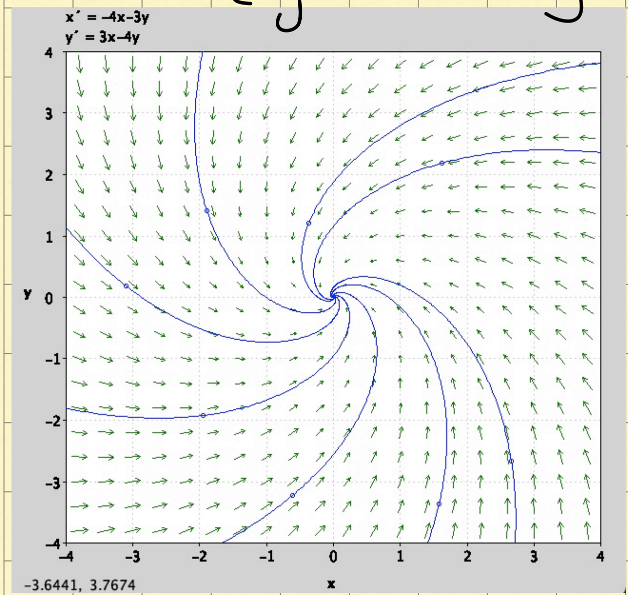
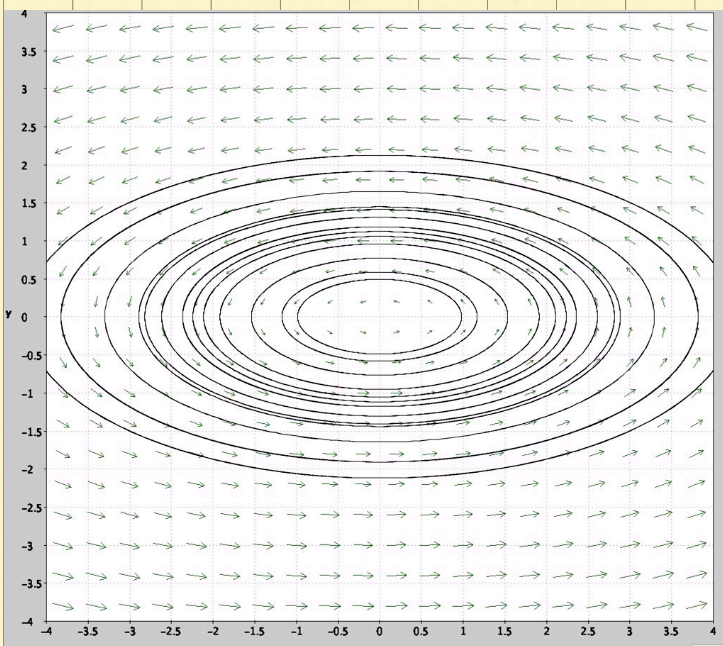


Ex: cplx e-values w/ 0 real part (purely imaginary).



trajectories ellipses, origin is a center.

Repeated Real Eigenvalues. (5.5)

Ex A: $\underset{=}{x}' = \underset{=}{A}\,\underset{=}{x}$ $\qquad \underset{=}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Need 2 lin. indep. sols.

$\det(\underset{=}{A} - \lambda \underset{=}{I})$

$= (\lambda - 1)^2$

$\Rightarrow \lambda = 1$ repeated e-value.

Look for evectors:

$$\overbrace{(\underset{=}{A} - 1 \cdot \underset{=}{I})}^{0}\, \underset{=}{v} = \underset{=}{0}$$

E.g. $\underline{\underline{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{\underline{v}}_2 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ are e-vectors.

$$\left.\begin{aligned} \underline{\underline{x}}_1(t) &= e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \underline{\underline{x}}_2(t) &= e^t \begin{bmatrix} 5 \\ -3 \end{bmatrix} \end{aligned}\right\} \text{lin. indep. sols.}$$

$$\underline{\underline{x}}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Had multiplicity 2 e-value, 2 lin. indep. e-vectors. //

Ex B.

$$\underline{\underline{x}}' = \underbrace{\begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}}_{\underline{\underline{A}}} \underline{\underline{x}}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \dots = (\lambda - 5)^2$$

$\Rightarrow \lambda = 5$ repeated root.

Find eigenvectors:

$$\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 = -v_2$ an e-vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

A sol'n of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ is $\underline{\underline{x}}(t) = e^{5t}\begin{bmatrix}1\\-1\end{bmatrix}$.

Q: How do we find a second lin. indep. sol'n?

A: Seek sol'n in a form different from $e^{\lambda t}\underline{\underline{v}}$.

Terminology: e-value, multiplicity k.
p associated e-values, $p \le k$
Defect is the number $k-p$.
If defect ≥ 1, say e-value defective.
Ex A: defect 0
B: defect 1.

Defect 1.

New guess: Sol'n of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ is

$$\underline{\underline{x}}(t) = e^{\lambda t}(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2)$$

unknown, const.

Plug in $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$

$$\underline{\underline{x}}'(t) = \underline{\underline{v}}_1 e^{\lambda t} + \lambda e^{\lambda t}(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) \quad \text{(pr. rule)}$$

$$\underline{\underline{v}}_1 \cancel{e^{\lambda t}} + \lambda \cancel{e^{\lambda t}}(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) = \underline{\underline{A}}\left(\cancel{e^{\lambda t}}(v_1 t + v_2)\right)$$

$$\Rightarrow (\lambda \underline{\underline{v}}_1 - \underline{\underline{A}}\,\underline{\underline{v}}_1)t + (\underline{\underline{v}}_1 + \lambda \underline{\underline{v}}_2 - \underline{\underline{A}}\,\underline{\underline{v}}_2) = 0$$

$$\left(\underline{\underline{B}}t + \underline{\underline{C}} = 0 \text{ for all } t \Rightarrow \underline{\underline{B}} = \underline{\underline{C}} = \underline{\underline{0}} \right)$$

$$\Rightarrow \begin{cases} (\underline{\underline{A}} - \lambda \underline{\underline{I}})\, \underline{\underline{v}}_1 = \underline{\underline{0}} \\ (\underline{\underline{A}} - \lambda \underline{\underline{I}})\, \underline{\underline{v}}_2 = \underline{\underline{v}}_1 \end{cases}$$

Notice: $(\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{\underline{v}}_2 = (\underline{\underline{A}} - \lambda \underline{\underline{I}})\, \underline{\underline{v}}_1 = \underline{\underline{0}}$

(check)

Method: λ e-value of defect 1

Find a $\underline{\underline{v}}_2 \neq 0$ so that

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{\underline{v}}_2 = \underline{\underline{0}}$$

and $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\, \underline{\underline{v}}_2 = \underline{\underline{v}}_1 \neq \underline{\underline{0}}$.

Then: $\underline{\underline{v}}_1 e^{\lambda t}$, $(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) e^{\lambda t}$ are a pair of lin. indep. sols.