Plan: Repeated eigenvalues of defect ≥ 1.

Last time: Given system $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$, λ eigenvalue of defect 1

Find a $\underline{\underline{v}}_2 \neq 0$ so that

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^{2} \underbrace{\underline{\underline{v}}_2}_{\text{generalized eigenvector}} = \underline{\underline{0}}$$

and

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_2 = \underbrace{\underline{\underline{v}}_1}_{\text{eigenvector.}} \neq \underline{\underline{0}}$$

$(A - \lambda I)\underline{\underline{v}}_1 = (A - \lambda I)^2 \underline{\underline{v}}_2 = \underline{\underline{0}}$

Then: $\underline{\underline{v}}_1 e^{\lambda t}$, $(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) e^{\lambda t}$ are a pair of linearly independent solutions for the system.

Ex:

$$\underline{\underline{x}}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \underline{\underline{x}}$$

$\lambda = 5$ is a repeated e-value of defect 1.

$$(\underline{\underline{A}} - 5\underline{\underline{I}}) = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}, \quad (\underline{\underline{A}} - 5\underline{\underline{I}})^2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} = \underline{\underline{0}}$$

Solve:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \overset{(\underline{\underline{A}} - \lambda \underline{\underline{I}})}{\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

e.g. take $v_1 = v_2 = 1$ (for example)

then $\underline{\underline{v}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{\underline{v}}_1 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$

Sols: $x_1 = \begin{bmatrix} -8 \\ 8 \end{bmatrix} e^{5t}$

$x_2 = \left(\begin{bmatrix} -8 \\ 8 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^{5t}$ are

a pair of lin. indep. sols; gen. sol'n

$$x = c_1 x_1(t) + c_2 x_2(t)$$

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Higher defect e-values

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (2-\lambda)^4$

$\Rightarrow \lambda = 2$ e-value of multiplicity 4.

Find eigenvectors:

$$\left(\underline{\underline{A}} - 2\underline{\underline{I}}\right)\underline{v} = \underline{\underline{0}} \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow u_2 = 0, \; u_3 = 0$

<u>So:</u>

$\begin{bmatrix} u_1 \\ 0 \\ 0 \\ u_4 \end{bmatrix}$, u_1, u_4 free to be chosen, are eigenvectors.

for example: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ pair of lin. indep. eigenvectors

<u>Any</u> eigenvector is of form

$u_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, so can't find 3rd lin. indep. e-vector.

Defect of $\lambda = 2$: $4 - 2 = 2$

<u>Terminology:</u>

Generalized e-vector of rank r, assoc. to e-value λ: a non-zero $\underline{\underline{v}}$ so that

$$\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)^{r} \underline{\underline{v}} = \underline{\underline{0}}, \quad \left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)^{r-1} \underline{\underline{v}} \neq \underline{\underline{0}}$$

Ex: $r = 1$ eigenvector.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in 1st example: gen. e-vector of rank 2.

We'll use chains of generalized e-vectors to build sol's.

Def'n: A length k chain of generalized e-vectors based on eigenvector $\underline{\underline{v}}_1$ is set of $\{\underline{\underline{v}}_1, \dots, \underline{\underline{v}}_k\}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\, \underline{\underline{v}}_k = \underline{\underline{v}}_{k-1}$$

$$\cdots$$

$$(A - \lambda I)\, \underline{\underline{v}}_3 = v_2$$

$$(A - \lambda I)\, \underline{\underline{v}}_2 = \underline{\underline{v}}_1$$

$\underline{\underline{v}}_k$ is rank k gen. e-vector.
If known we can find the chain.

In 1st example: Chain of length 2

$$\underline{\underline{v}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\, v_2 = \underline{\underline{v}}_1 = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

honest eigenvector.

To build sols to $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ $\underline{\underline{\text{if}}}$ given a chain of gen. eigenvectors:

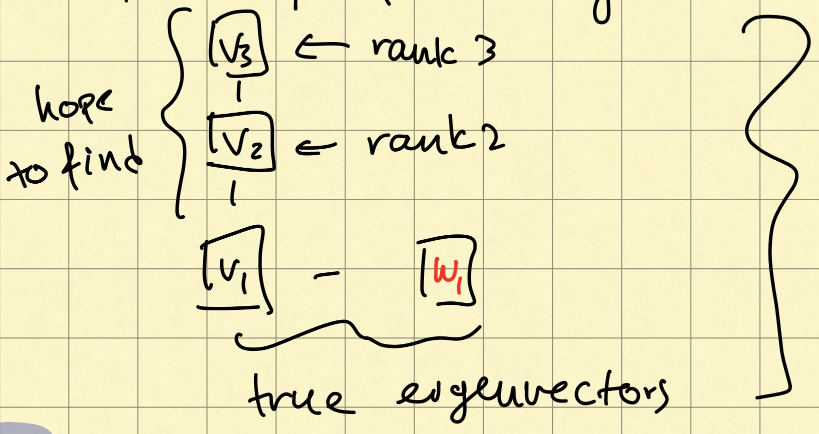
$$\left\{\begin{array}{l} x_1 = e^{\lambda t}\,\underline{\underline{v}}_1 \\ x_2 = e^{\lambda t}\left(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2\right) \\ \quad \dots \\ x_k = e^{\lambda t}\left(v_1 \dfrac{t^{k-1}}{(k-1)!} + v_2 \dfrac{t^{k-2}}{(k-2)!} + \dots + v_k \right) \end{array}\right.$$

are lin. indep. sols of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$

Go back to example:

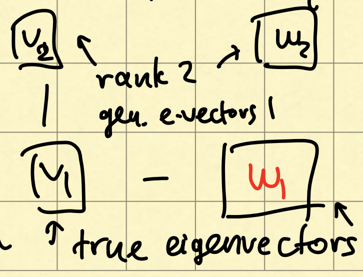
Multiplicity 4: found 2 eigenvectors.

Schematically:



if possible.

Another possible configuration:



Ex:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Goal: find $\underline{\underline{v}}_3$: gen. e-vector of $\underbrace{\text{rank}}_{\text{defect} + 1}$ 3

If $\underline{\underline{v}}_3$ is a gen. e-vector of rank 3 then:

$$\left\{\begin{array}{l}(A-\lambda I)^3 \underline{\underline{v}}_3 = \underline{\underline{0}}\\ (A-\lambda I)^2 \underline{\underline{v}}_3 \neq \underline{\underline{0}}\end{array}\right.$$

$$(A-\lambda I) \underline{\underline{v}}_3 \neq \underline{\underline{0}}$$

Want:

$$(\underline{\underline{A}} - 2\underline{\underline{I}})^3 \underline{\underline{v}}_3 = \underline{\underline{0}}$$

$$(\underline{\underline{A}} - 2\underline{\underline{I}})^2 \underline{\underline{v}}_3 \neq 0.$$

Check:

$$(\underline{\underline{A}} - 2\underline{\underline{I}})^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\underline{\underline{A}} - 2\underline{\underline{I}})^3 = \underline{\underline{0}}$$

Want: v_3 : $(A-2I)^3 \underline{\underline{v}}_3 = 0$

$$(A-2I)^2 \underline{\underline{v}}_3 \neq 0$$

Ex: $\underline{\underline{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = (A-2I)\underline{\underline{v}}_3$

$$= \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{v}}_1 = (A - 2I)\,\underline{\underline{v}}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{an eigenvector.}}}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

<u>recap / finish on Friday.</u>