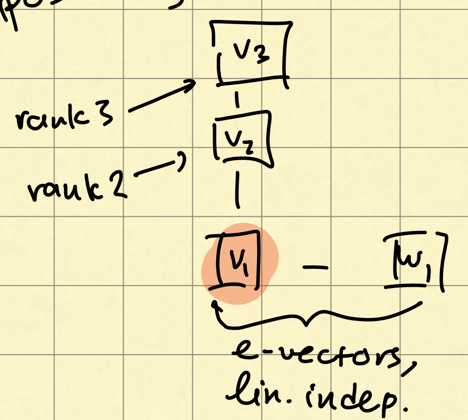
Plan: 1. Recap high defect method from last time (5.5).
2. See phase plane portraits in case of defective eigenvalues (5.3)
3. Start discussing autonomous systems. (6.1)

System: $\underline{x}' = \underline{A}\,\underline{x}$ $\underline{A}$ has eigenvalue(s) of high defect.

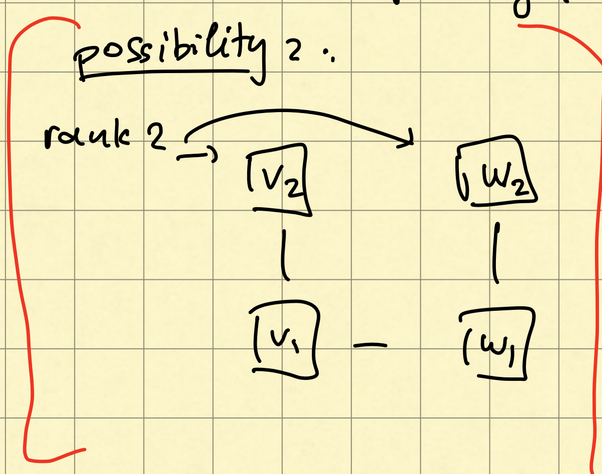
Idea: if λ has defect $k \geq 1$ use chains of gen. e-vectors to "fill up" multiplicity

Ex: Multiplicity 4, defect 2, i.e. only 2 lin. indep. eigenv.

possibility 1:



possibility 2:



won't appear in HW

Method: 1. Find eigenvalues, eigenvectors to compute defect.

2. If λ has defect k: try to find a chain of length $(k+1)$ starting at a gen. e-vector of rank $k+1$.

Try to solve, if possible.

$$\left.\begin{aligned}(A-\lambda I)^{k+1} u_{k+1} &= \underline{\underline{0}}\\ (A-\lambda I)^{k} u_{k+1} &\neq \underline{\underline{0}}\end{aligned}\right\} \circledast$$

If u_{k+1} exists, build chain

$$(A-\lambda I)\, u_{k+1} = u_k$$

$$\cdots$$

$$(A-\lambda I)\, u_2 = u_1 \rightarrow \text{true eigenv.}$$

3. "Fill up" multiplicity using chains based on eigenvectors lin. indep. from u_1.

4. $\circledast$ might not be possible (e.g. if $(\underline{\underline{A}} - \lambda \underline{\underline{I}})^k = \underline{\underline{0}}$).
In that case try to find chain of length k etc.

ex: $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5. Once found as many gen. eigenvectors as the multiplicity, build sol's as described on Wednesday.

Ex. from last time.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Found: $\lambda = 2$ has mult. 4.

$\begin{bmatrix} a \\ 0 \\ 0 \\ b \end{bmatrix}$ eigenv. for all a, b.

$\Rightarrow$ at most 2 lin. indep. e-vectors.

Defect is 2. Try for chain of length 3.

Found:

$$(\underline{\underline{A}} - 2\underline{\underline{I}})^3 v_3 = \underline{\underline{0}}$$
$$(\underline{\underline{A}} - 2\underline{\underline{I}})^2 v_3 \neq \underline{\underline{0}} \quad \text{for} \quad \underline{\underline{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

So,

$$\begin{tikzcd} \boxed{v_3} \arrow[d, dash] & \\ \boxed{v_2} \arrow[d, dash] & \\ \boxed{v_1} \arrow[r, dash] & \boxed{w_1} \end{tikzcd}$$

Build chain:

$$\underline{\underline{v}}_2 = (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{v}}_1 = (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Can take $w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ to "fill up" multiplicity

4 Sols:

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}, \quad \tilde{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{2t}$$

$$\tilde{x}_2 = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) e^{2t}$$

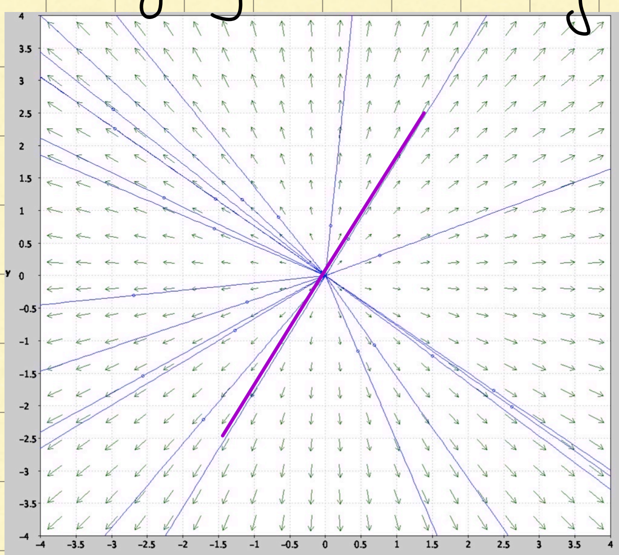
$$\tilde{x}_3 = \left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\frac{t^2}{2} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} t + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) e^{2t} \qquad //$$

Phase plane portraits

$\underset{=}{x}' = \underset{=}{A}\,\underset{=}{x}$, $\quad \underset{=}{A}$ 2×2 matrix

Ex 1 defect 0, positive repeated e-values.

$$x' = x \qquad x = ae^t$$
$$y' = y \qquad y = be^t$$



traj. are half lines starting at origin. recede from origin

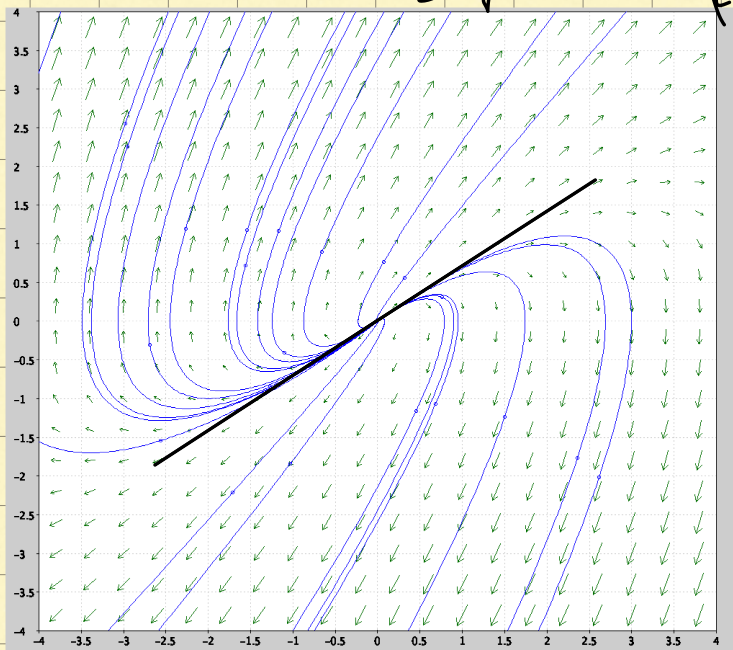
proper nodal source

at most one pair of "opposite" trajectories tangent to the same line at origin.

Ex 2 Repeated positive e-v. defect 1.

$$x' = y$$
$$y' = -x + 2y$$

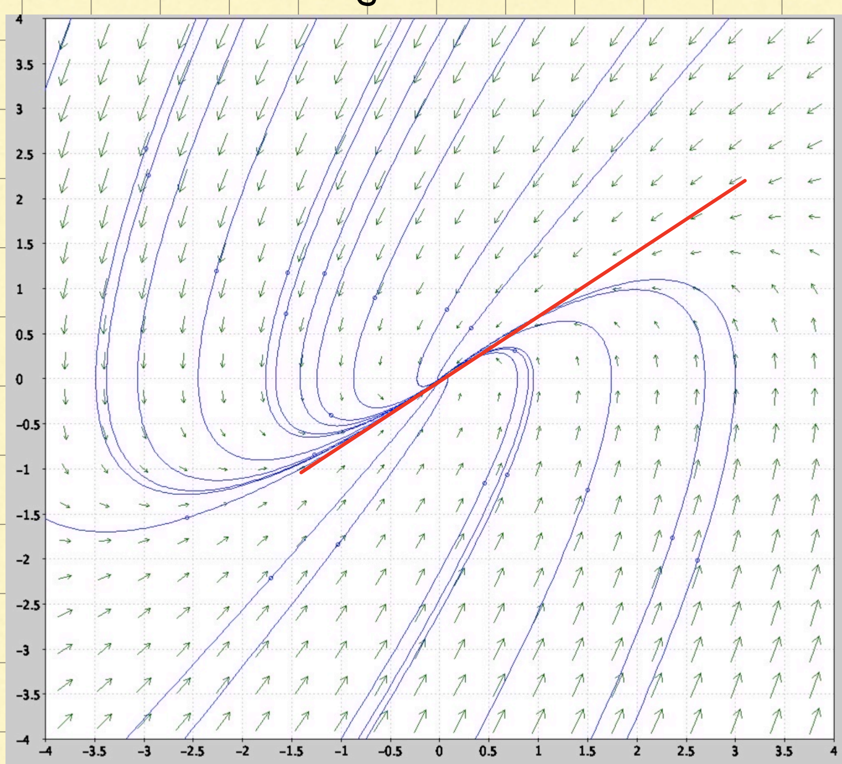
$$\underline{\underline{x}}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) e^{t}$$



All traj. tang. to a line at origin, recede from origin.

improper nodal source.

Ex 3: negative repeated e-values, defect 1.



improper nodal sink

Look at p. 315-316 in textbook.

Ch 6. Non-linear systems.

6.1 Autonomous systems.

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

no time dependence.

Ex: $\begin{cases} \dfrac{dx}{dt} = 2x+y \\ \dfrac{dy}{dt} = y \end{cases}$ linear, autonomous.

$\begin{cases} \dfrac{dx}{dt} = \sin(y) \\ \dfrac{dy}{dt} = \cos(x) \end{cases}$ non-linear, autonomous.

Non-Ex: $\begin{cases} \dfrac{dx}{dt} = 2x+y+t \\ \dfrac{dy}{dt} = x+2y+\cos(t) \end{cases}$ non-autonomous linear (non-homogeneous)

Existence & Uniqueness

If (x_0,y_0), t_0 given, F, G are nice then there is exactly one sol'n of $x'=F(x,y)$, $y'=G(x,y)$ in an interval containing t_0 so that $\begin{cases} x(t_0)=x_0 \\ y(t_0)=y_0 \end{cases}$

<u>Fact:</u> If (x_0, y_0) is such that $\begin{cases} F(x_0,y_0)=0 \\ G(x_0,y_0)=0 \end{cases}$

<u>then:</u> $\begin{cases} x(t) = x_0 \\ y(t) = y_0 \end{cases}$ is a sol'n to

$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$. Such an (x_0,y_0) is called a <u>critical point</u> for the system.