Plan: - Discuss critical pts, equilibrium sol's for autonomous systems
- Stability/ asymptotic stability, classification of critical pts.
- "Moving" critical points to the origin
- Linearizing non-linear systems.

Autonomous Systems

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

not dependence.

A pt (x_0,y_0) is called a critical point (CP) if

$$\begin{cases} F(x_0,y_0) = 0 \quad \text{and} \\ G(x_0,y_0) = 0 \end{cases}$$

Ex: $\begin{cases} x' = \sin(y) \\ y' = \cos(x) \end{cases}$

Find CP: $\sin(y) = 0 \Rightarrow y = k\pi$, k integer

$\cos(x) = 0 \Rightarrow x = m\pi + \frac{\pi}{2}$, m integer

∞ many CP. //

Ex: $\frac{dx}{dt} = 2 - 4x - 15y$

$\frac{dy}{dt} = 4 - x^2$

Find CP: $\begin{cases} 2 - 4x - 15y = 0 \\ 4 - x^2 = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{1}{15}(2-4x) \\ x = \pm 2 \end{cases}$

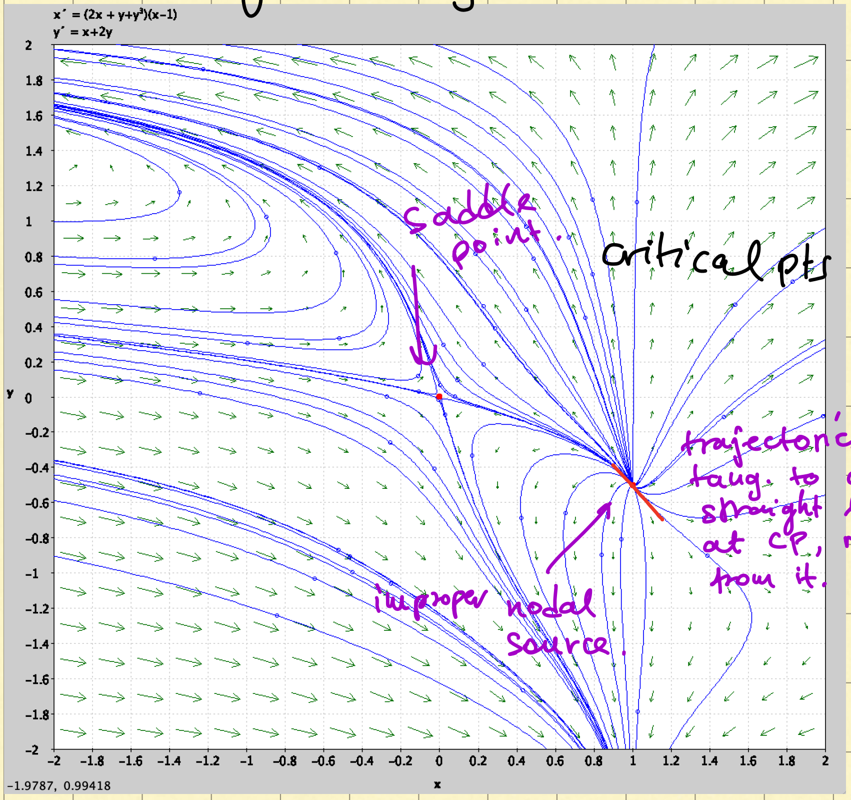
$\Rightarrow \left(2, -\frac{6}{15}\right), \left(-2, \frac{10}{15}\right)$ //

CP give locations of equilibrium sols:

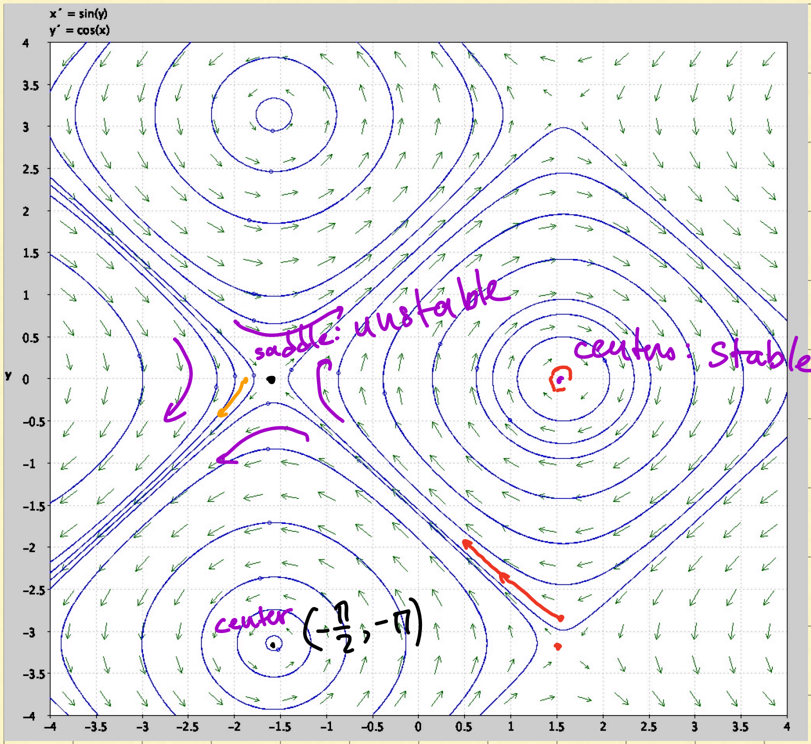
If (x_0, y_0) is a CP then $\begin{cases} x(t) = x_0 \\ y(t) = y_0 \end{cases}$ is a sol'n to the system.

Use phase plane portraits to understand non-const. sols.

Ex: $\begin{cases} x' = (2x+y+y^3)(x-1) \\ y' = x+2y \end{cases}$



Talk of the CP as being saddles/nodes/spirals etc.



$x' = \sin(y)$
$y' = \cos(x)$

Found: CPs

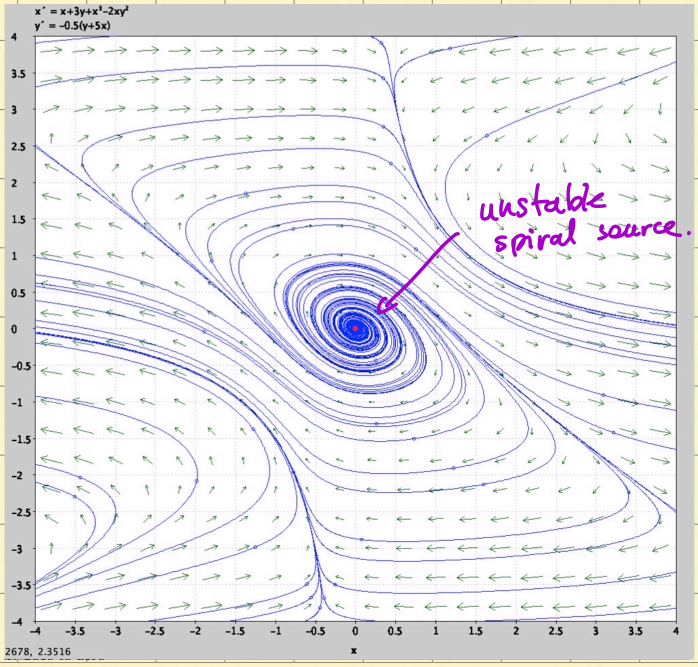
$(k\pi + \frac{\pi}{2}, m\pi)$

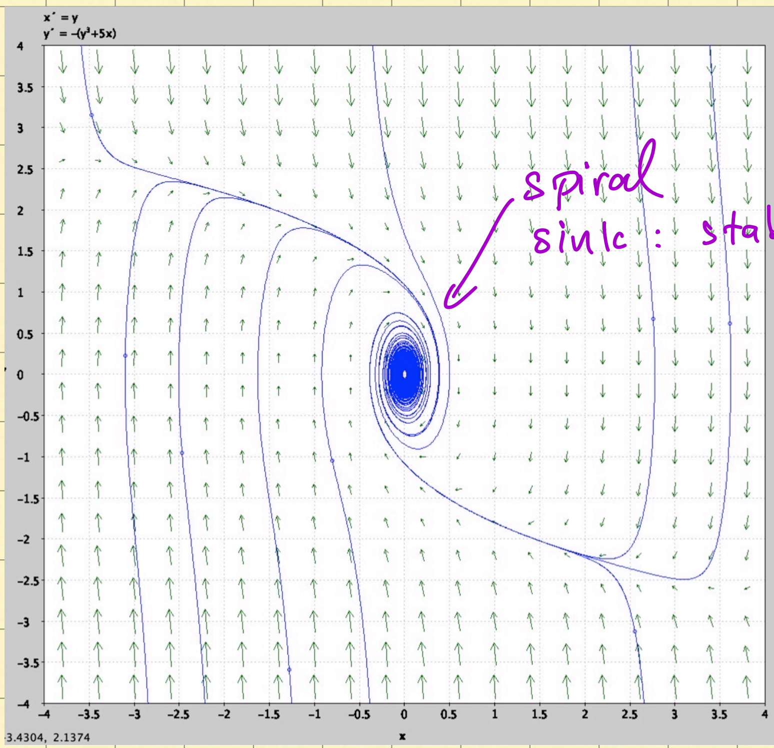
k, m integers.

Stable / Unstable CP

Stable CP: every solution which starts suf. close to the CP stays close to the CP.

otherwise: unstable CP.





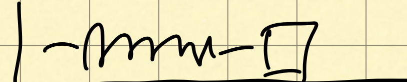
spiral sink : stable , asymptotically stable.

A CP is called asymptotically stable if every sol'n which starts near the CP approaches it as $t \to \infty$

spiral sinks, nodal sinks   asymptotically stable

centers: stable, not asymptotically stable.

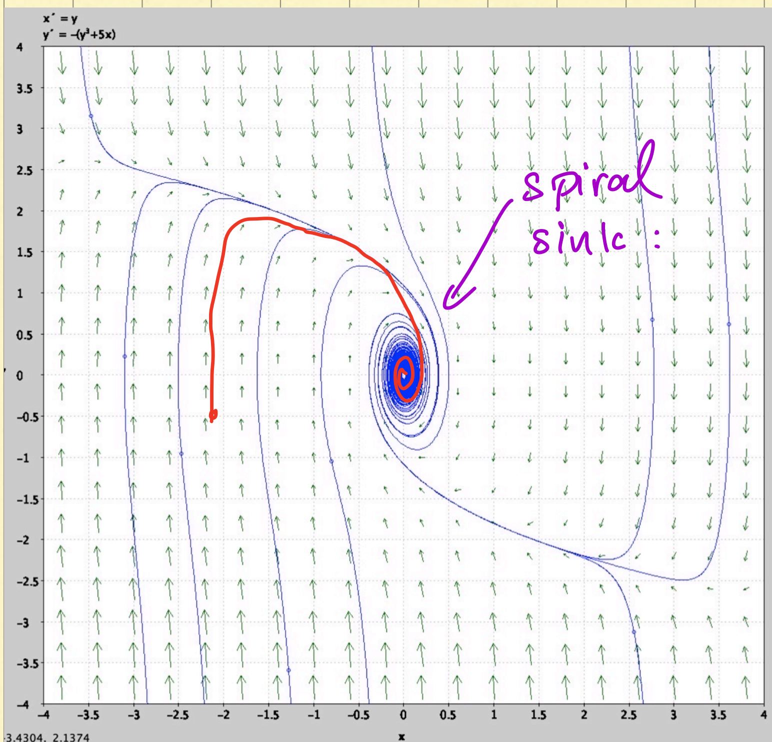
Ex: mechanical spring-mass systems



$$\underset{\text{mass}}{m}x'' + \underbrace{c(x')^3}_{\text{damping}} + \underset{\text{spring const.}}{k}x = 0$$

non-linear

set: $x_1 = x$
$x_2 = x'$ $\Rightarrow$ $\begin{cases} x_1' = x_2 \\ x_2' = \underbrace{-\frac{1}{m}\left(c\, x_2^3 + k x_1\right)}_{\text{autonomous}} \end{cases}$



$\uparrow$ x_2, velocity.

traj. approach equil as $t \to \infty$.

$\longrightarrow$ x_1, displacement from equil

6.2 Analyze non-linear systems near their CP

Def'n: A CP is isolated if there is a neighborhood of it which contains no other CP.

Ex: $x' = \sin(y)$ $\quad (k\pi + \frac{\pi}{2}, m\pi)$ k, m int
$y' = \cos(x)$

Non-ex: $\left.\begin{array}{l} x' = x \\ y' = x \end{array}\right\}$ all of y axis is CP

If given an isolated CP (x_0, y_0), we can use change of variables $\begin{cases} u = x - x_0 \\ v = y - y_0 \end{cases}$ to obtain an equivalent system w/ an isolated CP at the origin

Ex: $\begin{cases} \dfrac{dx}{dt} = 2x - 2y - 4 \\ \dfrac{dy}{dt} = x + 4y + 3 \end{cases}$

CP: $\begin{cases} 2x - 2y - 4 = 0 \\ x + 4y + 3 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}$ only one CP, isolated.

Set: $u = x - 1 \quad \rightarrow x = u + 1$
$v = y + 1 \quad \Rightarrow y = v - 1$

$$\begin{cases} \dfrac{du}{dt} = 2(u+1) - 2(v-1) - 4 \\ \dfrac{dv}{dt} = (u+1) + 4(v-1) + 3 \end{cases}$$

$$\Rightarrow \left.\begin{cases} \dfrac{du}{dt} = 2u - 2v \\ \dfrac{dv}{dt} = u + 4v \end{cases}\right\} \text{CP at origin, isolated.}$$

Reminder:

Taylor's formula for functions of 2 variables.

$f(x,y)$ nice

Can write, given (x_0,y_0)

$$f(x_0+u,\, y_0+v) = \underbrace{f(x_0,y_0)}_{\text{const}} + \overbrace{\partial_x f\big|_{(x_0,y_0)} u + \partial_y f\big|_{(x_0,y_0)} v}^{\text{linear}} + \overbrace{r(u,v)}^{\text{error}}$$

think of as small

where $\underbrace{\lim_{(u,v)\to(0,0)} \frac{r(u,v)}{\sqrt{u^2+v^2}} = 0}_{}$

error small relative to $|(u,v)|$ for small $|(u,v)|$.