Sketch of solutions.

1. a) Eigenvalues: $\lambda = 2 \pm 2i$

Eigenvector eq'n:

$$\begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Take $\underline{\underline{v}} = \begin{bmatrix} -5 \\ 1+2i \end{bmatrix}$

<u>Sol'n:</u>

$$\underline{\underline{x}}(t) = \underline{v}\, e^{(2+2i)t}$$

$$= e^{2t} \begin{bmatrix} -5\cos(2t) - 5i\sin(2t) \\ \cos(2t) - 2\sin(2t) + i\left(\sin(2t) + 2\cos(2t)\right) \end{bmatrix}$$

Take real & imaginary parts: sols:

$$\underline{\underline{x}}_1(t) = e^{2t} \begin{bmatrix} -5\cos(2t) \\ \cos(2t) - 2\sin(2t) \end{bmatrix}$$

$$\underline{\underline{x}}_2(t) = e^{2t} \begin{bmatrix} -5\sin(2t) \\ \sin(2t) + 2\cos(2t) \end{bmatrix}$$

Check linear indep.: Wronskian

$$W(x_1, x_2) = e^{4t} \begin{vmatrix} -5\cos(2t) & -5\sin(2t) \\ \cos(2t) - 2\sin(2t) & \sin(2t) + 2\cos(2t) \end{vmatrix}$$

$$= e^{4t}\Big(-5\cancel{\cos(2t)\sin(2t)} - 10\cos^2(2t)$$

$$+5\cancel{\sin(2t)\cos(2t)} - 10\sin^2(2t)\Big)$$

$$= -10e^{4t} \neq 0 \Rightarrow \text{lin. indep.}$$

b. Fig. 1 (spiral source).

2. Eigenvalues: $\lambda_1 = -2$, $\lambda_2 = 5$

Eigenvectors:

for $\lambda_1 = -2$, $\underline{v}_1 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

for $\lambda_2 = 5$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

gen. sol'n: $\underline{x} = c_1 e^{-2t}\begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underline{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 2 \\ -6c_1 + c_2 = -1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{3}{7} \\ c_2 = \frac{11}{7} \end{cases}$$

$$\Rightarrow \underline{x}(t) = \frac{3}{7}e^{-2t}\begin{bmatrix} 1 \\ -6 \end{bmatrix} + \frac{11}{7}e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. Find defect: Eigenvector system $(\underline{A} - 2\underline{I})\underline{v} = \underline{0}$

$$\begin{bmatrix} -17 & -7 & 4 \\ 34 & 14 & -11 \\ 17 & 7 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -17 & -7 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -17 & -7 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad v_3 = 0$

$-17 v_1 - 7 v_2 + 4 v_3 = 0 \Rightarrow$

$v_1 = -\frac{7}{17} v_2$

So defect 2. Look for chain of length 3.

Look for gen. e-vector of rank 3: want

$(A - \lambda I)^3 \underline{\underline{v}}_3 = 0$

given $\Rightarrow$ $(A - \lambda I)^2 \underline{\underline{v}}_3 \neq 0$

Take $\underline{\underline{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Then $\underline{\underline{v}}_2 = (A - \lambda I) \underline{\underline{v}}_3 = \begin{bmatrix} 4 \\ -11 \\ 3 \end{bmatrix}$

$\underline{\underline{v}}_1 = \begin{bmatrix} 21 \\ -51 \\ 0 \end{bmatrix}$ $\leftarrow$ true eigenvector

Soln:

$$\underline{\underline{x}} = c_1 e^{2t}\begin{bmatrix}21\\-51\\0\end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix}21\\-51\\0\end{bmatrix}t + \begin{bmatrix}4\\-11\\3\end{bmatrix}\right)$$

$$+ c_3 e^{2t}\left(\begin{bmatrix}21\\-51\\0\end{bmatrix}\frac{t^2}{2} + \begin{bmatrix}4\\-11\\3\end{bmatrix}t + \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

4. a). $F(0,0) = G(0,0) = 0$, so $(0,0)$ is crit. pt

b). Jacobian:

$$J = \begin{bmatrix} y e^{x+y} & (y+1)e^{x+y} \\ -(x+1)e^{x+y} & -x e^{x+y} \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$ Eigenvalues $\lambda = \pm i$

Origin is a stable center.

c). The C.P. $(0,0)$ for the non-linear system is a stable center, a spiral source or a spiral sink.

d). $$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow y\,dy = -x\,dx$$

$$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$\Rightarrow x^2 + y^2 = 2C$$

so the origin is a center for the non-linear system as well.

5. Char. eq'n: $(4-\lambda)(-1-\lambda) - \varepsilon = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 - \varepsilon = 0$

$\Rightarrow \lambda = \frac{1}{2}\left(3 \pm \sqrt{25 + 4\varepsilon}\right)$

Unstable node: eigenvalues must be both real & positive (distinct or repeated), so we want

1. $25 + 4\varepsilon \geq 0 \Rightarrow \varepsilon \geq -\frac{25}{4}$

2. $\sqrt{25+4\varepsilon} < 3$

$\Rightarrow 25 + 4\varepsilon < 9 \Rightarrow$

$\Rightarrow \varepsilon < -4.$

So $-\frac{25}{4} \leq \varepsilon < -4$

6. a) Predation: x is the predator

y is the prey.

b) Want $x, y \geq 0$

$$\begin{cases} 7x - x^2 + xy = 0 & (1) \\ y - 4xy = 0 & (2) \end{cases}$$

(2) $\Rightarrow$ $y = 0$ or $x = \frac{1}{4}$

If $y = 0$ then $7x - x^2 = 0 \Rightarrow x = 0$ or $x = 7$

If $x = \frac{1}{4}$ then ① $\Rightarrow$ $7 - \frac{1}{4} + y = 0 \Rightarrow$

$y = -7 + \frac{1}{4} < 0$

not relevant.

So $(0,0)$, $(7,0)$ are the only relevant ones.

c). The population of the prey eventually vanishes and the predators approach $x = 7$.

7. Set $y = x'$. Then:

$$\begin{cases} x' = y \\ y' = -4x + 5x^3 - x^5 \end{cases}$$

Equil: find C.P.

$$\begin{cases} y = 0 & ① \\ -4x + 5x^3 - x^5 = 0 & ② \end{cases}$$

② $\Rightarrow$ $x = 0$ or

$-4 + 5x^2 - x^4 = 0 \Rightarrow x^4 - 5x^2 + 4 = 0$

$\Rightarrow x^2 = \frac{1}{2}(5 \pm 3) \Rightarrow x^2 = 4$ or $x^2 = 1$

$\Rightarrow x = \pm 1$, $x = \pm 2$.

So C.P. $(0,0), (\pm 2, 0), (\pm 1, 0)$. For each $(x(t), y(t)) = (x_0, y_0)$ is an equil. sol'n.

8. Do it for general a:

$$\int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{s-a}, \quad s > a \qquad \text{(done in class)}$$

So $\mathcal{L}\{\sinh(t)\} = \frac{1}{2}\mathcal{L}\{e^{t}\} - \frac{1}{2}\mathcal{L}\{e^{-t}\} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$

$$= \frac{1}{s^2-1}$$

for $s > 1$

9. Taking Laplace:

$$sX(s) - 1 = 2X(s) + Y(s)$$
$$sY(s) + 2 = 6X(s) + 3Y(s)$$

$$\Rightarrow \begin{cases} X(s) = \frac{1}{s} \\ Y(s) = -\frac{2}{s} \end{cases}$$

So $\begin{cases} x(t) = 1 \\ y(t) = -2 \end{cases}$

10. Full sol'n of this can be found in the handout on chains of gen. e-vectors, Ex. 2.

https://www.math.purdue.edu/~neptamin/303Au21/Handouts/High_defect.pdf

b) Check that $A\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\3\end{bmatrix}$

c) We'd have to solve

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad 0 \cdot d = 1$, impossible.

So the bottom-to-top approach does not work in this case.