Problem 1:

a) $\lambda = 0$:

$$\Theta'' = 0 \Rightarrow \Theta = A + B\theta$$

So $\Theta = A$.

b) $\lambda = -a^2$

$$\Theta'' = a^2\Theta \Rightarrow \Theta = A\cosh(a\theta) + B\sinh(a\theta)$$

$$\Rightarrow A = B = 0$$

c) $\lambda = a^2 > 0$

$$\Theta'' + a^2\Theta = 0 \Rightarrow \Theta = A\cos(a\theta) + B\sin(a\theta)$$

Has non-trivial 2π-periodic sol'n when $a = n$, a positive integer.

So: $\lambda = 0$ or $\lambda = n^2$, n integer.

Problem 2:

$$r^2R'' + rR = 0$$

$u = R' \Rightarrow r^2u' + ru = 0 \Rightarrow \frac{u'}{u} + \frac{1}{r} = 0$

$\Rightarrow \ln u + \ln r = C \Rightarrow \ln(ur) = C$

$\Rightarrow ur = B \Rightarrow$

$\Rightarrow R' = \frac{B}{r} \Rightarrow R(r) = A + B\ln r.$

Problem 3:

Set $R(r) = r^k$

$$r^2 k(k-1) r^{k-2} + r k r^{k-1} - n^2 r^k = 0$$

$$\Rightarrow \quad k(k-1) + k - n^2 = 0$$

$$\Rightarrow k^2 - n^2 = 0 \Rightarrow k = \pm n$$

So

$$R(r) = A r^n + B r^{-n}$$

Steps.

1.

$$R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0$$

$$\Rightarrow r^2 \frac{R''}{R} \Theta + r \frac{R'}{R} \Theta + \Theta'' = 0$$

$$\Rightarrow \left(r^2 \frac{R''}{R} + r \frac{R'}{R} \right) = -\frac{\Theta''}{\Theta}$$

2. So: $-\frac{\Theta''}{\Theta} = \lambda \Rightarrow \Theta'' + \lambda \Theta = 0.$ ✓

Since $u_n(r, \theta) = u_n(r, \theta + 2\pi)$

$$\Rightarrow R(r)\Theta(\theta) = R(r)\Theta(\theta + 2\pi)$$

$$\Rightarrow \Theta(\theta) = \Theta(\theta + 2\pi)$$ ✓

3. So $\lambda = 0$ or $\lambda = n^2$

4. Thus either $r^2 R_0'' + r R_0' = 0$ or

$$r^2 R_n'' + r R_n' - n^2 R_n = 0$$

So $\left.\begin{array}{l} R_0(r) = A + B\ln r \\ R_0(\beta) = 0 \end{array}\right\} \Rightarrow$ $A + B\ln\beta = 0$

$\Rightarrow A = -B\ln\beta$

$\Rightarrow R_0(r) = -B(\ln\beta + \ln r) = -B\ln\frac{r}{\beta}$

$\left.\begin{array}{l} R_n(r) = Ar^n + Br^{-n} \\ R_n(\beta) = 0 \end{array}\right\} \Rightarrow$ $A\beta^n + B\beta^{-n} = 0$

$\Rightarrow A = -\beta^{-2n}B$

$\Rightarrow R_n(r) = B\left(-\beta^{-2n}r^n + r^{-n}\right) = B\left(-\left(\frac{r}{\beta}\right)^n + \left(\frac{r}{\beta}\right)^{-n}\right)$

5. So: take

$u_0 = \ln\frac{r}{\beta}$

$u_n = \left(\left(\frac{r}{\beta}\right)^n - \left(\frac{r}{\beta}\right)^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$

So:

$$u(r,\theta) = A_0\ln\frac{r}{\beta} + \sum_{n=1}^{\infty}\left(\left(\frac{r}{\beta}\right)^n - \left(\frac{r}{\beta}\right)^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

6. $u(\alpha,\theta) = A_0\ln\frac{\alpha}{\beta} + \sum_{n=1}^{\infty}\left(\left(\frac{\alpha}{\beta}\right)^n - \left(\frac{\alpha}{\beta}\right)^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$

$= f(\theta)$.

Take $A_0\ln\frac{\alpha}{\beta} = \frac{a_0}{2}$, where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$

$$A_n\left(\left(\frac{\alpha}{\beta}\right)^n-\left(\frac{\alpha}{\beta}\right)^{-n}\right)=a_n, \quad a_n=\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n\left(\left(\frac{\alpha}{\beta}\right)^n-\left(\frac{\alpha}{\beta}\right)^{-n}\right)=b_n, \quad b_n=\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta.$$