Applications of Fourier series

In this worksheet we will use Fourier series to explore the response of spring-mass systems (damped or undamped) to periodic external forces.

Undamped motion

The equation for the displacement x from equilibrium of an undamped spring-mass system under the influence of an external force $F(t)$ has the form

$$mx'' + kx = F(t), \tag{1}$$

where m denotes the mass and k the spring constant. Write $\omega_0 = \sqrt{k/m}$ for the *natural frequency* of the system. The general solution is of the form

$$s(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + x_p(t),$$

where $x_p(t)$ is a particular solution which depends on the force $F(t)$. Note that the first two terms are independent of the extenal force: they only depend on the parameters of the system and the initial conditions.

Assume that $F(t)$ is piecewise smooth and periodic with period $2L$ **and that** $\omega_0 \neq \pi n/L$ **for every integer** n. Then we can write a Fourier series for F and use it to find a particular solution of (1) which is periodic of period $2L$, in the form

$$x_{\mathrm{sp}}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right). \tag{2}$$

We call the formal solution obtained this way a *steady periodic solution*.

Problem 1. *Suppose that $m = 1$, $k = 4$, and that the external force $F(t)$ is given by the even function of period $2L = 4$ such that $F(t) = 2t$ if $0 < t < 2$.*

1. *Sketch the graph of $F(t)$ for $-6 < t < 6$.*
2. *Compute the Fourier series of $F(t)$*
3. *Plug into* (1) *the Fourier series you found in Part 2 and the Fourier series for the steady periodic solution in* (2) *to determine the coefficients a_n and b_n. In this way you can find the Fourier series expansion of the steady periodic solution.*
4. *Write the general solution of* (1) *with for the specific parameters and F given in this problem.*

Remark 1. *As you found in Problem 1, if the equation has the form in* (1) *(with no term of the form cx' in the left hand side), then if the Fourier series of F has no sine terms, the same holds for the Fourier series* (2) *of x_{sp}. Similarly, if the Fourier series of F has no cosine terms, the same holds for the one of x_{sp}.*

Pure resonance

We assumed before that the period of the external force was such that $\omega_0 \neq n\pi/L$ for all positive integers n. If there is any positive integer for which $\omega_0 = n\pi/L$ then we have the phenomenon of *pure resonance*, which means that the amplitude of the general solution $x(t)$ increases unboundedly.

Problem 2. *Let $m = 1$, $k = \pi^2/4$ in* (1)*, with F as in Problem 1. As you already found out, if there is a periodic solution with period $2L = 4$ with Fourier series expansion* (2)*, it will not have sine terms. Assuming that such a Fourier series exists, try to determine a_0, a_1 and a_2. What goes wrong when you try to find a_1?*

If the Fourier series of F has the expansion

$$F(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right) \tag{3}$$

and there is a positive integer N for which $\frac{N\pi}{L} = \omega_0 = \sqrt{k/m}$ then the solution $x(t)$ analogous to (2) is of the form

$$x(t) = -\frac{b_N}{2m\omega_0} t\cos(\omega_0 t) + \sum_{n\neq N} \frac{b_n}{m(\omega_0^2 - n^2\pi^2/L^2)} \sin\left(\frac{n\pi t}{L}\right). \tag{4}$$

Notice that the first term increases unboundedly and the sine term with the "problematic" frequency $N\pi/L$ is not included in the right hand side.[1] Eq. (4) is *not* a Fourier series.

Problem 3. *Determine whether pure resonance occurs for the following combinations of parameters m, k and $F(t)$:*

1. $m = 1$, $k = 4\pi^2$ *and $F(t)$ is the odd function of period 2 with $F(t) = 2t$ for $0 < t < 1$*
2. $m = 2$, $k = 10$; *$F(t)$ is the odd function of period 2 with $F(t) = 1$ for $0 < t < 1$.*

Bonus: if pure resonance occurs, write a solution in the form (4) *and plot the first few terms using a computer algebra system.*

Damped forced motion

In the presence of damping, the equation of motion for the spring mass system under the influence of a periodic external force $F(t)$ becomes

$$mx'' + cx' + kx = F(t), \quad c > 0. \tag{5}$$

The solution to (5) consists of a sum of a steady periodic solution and a transient solution, which decays exponentially fast. If $F(t)$ has an expansion of the form (3), then the steady periodic solution has the expansion

$$x_{\text{sp}}(t) = \sum_{n=1}^{\infty} \frac{b_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}},$$

where $\omega_n = n\pi/L$ and the phase angle α_n is the angle satisfying

$$\tan\alpha_n = \frac{c\omega_n}{k - m\omega_n^2} \text{ and } 0 < \alpha_n < \pi.$$

Problem 4. *If $m = 2$, $c = 0.01$, $k = 4$, and $F(t)$ is the force in Problem 3 p.2, determine the coefficients and phase angles for the first three nonzero terms of the series corresponding to x_{sp}.*

[1] To see how the solution (4) was derived, you can think of using undetermined coefficients to solve infinitely many differential equations of the form $mx'' + kx = b_n \sin(n\pi t/L)$ and superimposing them. For $n = N$, your solution will look like the first term in (4).