Lesson 1. 01/10/2022

Plan: Intro to linear systems
Write linear systems in matrix form ?

Fig. 1.

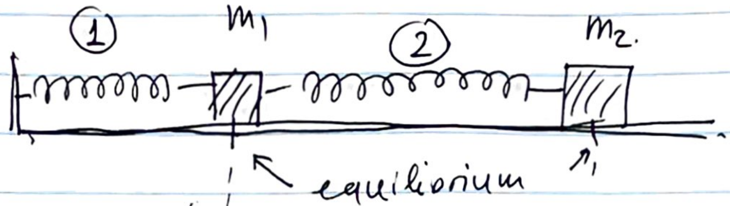
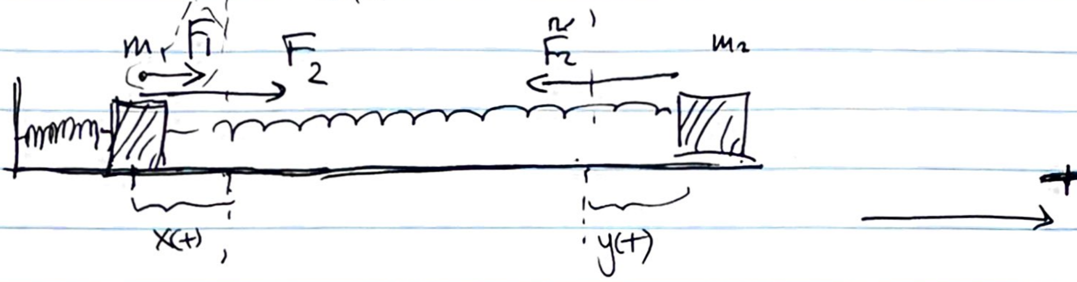


Fig. 2



$x(t)$: displacement of m_1 from equil.
In Fig 2: $x(t) < 0$.
$y(t)$: displacement of m_2
In Fig 2: $y(t) > 0$.

Goal: find $x(t)$, $y(t)$ by setting up differential eqs and solving them.

Hooke's Law:

F_1 : force exerted on m_1 by spring (1).

$F_1 = -k_1 X$

k_1: const. specific to spring (1).

F_2: force exerted on m_1 by spring (2).

$$F_2 = \underset{?}{+}\, k_2 \underbrace{(y(t) - x(t))}_{\text{stretching of spring (2)}}$$

k_2: ~~force~~ const. specific to spring (2).

$\tilde{F}_2$: force exerted on m_2 by spring (2).

$\tilde{F}_2 = -F_2 = -k_2(y(t) - x(t))$

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By Newton's law:

$$(*)\begin{cases} m_1 x'' = \underbrace{-k_1 x}_{F_1} + \underbrace{k_2(y-x)}_{F_2} \\ m_2 y'' = \underbrace{-k_2(y-x)}_{\tilde{F}_2} \end{cases}$$

System of ordinary dif'l eq's (ODE): derivatives wrt only one variable appear.
Order of system is 2: highest order of derivative is 2.

Now: we will turn $(*)$ into an equivalent system of order 1.

Set: $u_1 = x$ $\quad v_1 = y$
$u_2 = x'$ $\quad v_2 = y'$

Rewrite $(*)$:

$$\begin{cases} m_1 u_2' = -k_1 u_1 + k_2(v_1 - u_1) \\ m_2 v_2' = -k_2(v_1 - u_1) \\ u_1' = u_2 \\ v_1' = v_2 \end{cases}$$

System of 4 ODEs of order 1, equivalent to $(*)$.
Note: reduced the order at the expense of having more eq's.

From now on: work w/ 1st order systems.

<u>Review on matrices</u>: §5.1 in textbook, also see summary in today's calendar entry.

Matrix Valued Function

$$\underline{\underline{A}}(t) = \underbrace{\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \vdots & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{bmatrix}}_{n \text{ columns.}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \end{bmatrix}}\right\} m \text{ rows}$$

<u>Ex:</u> $\underline{\underline{A}}(t) = \begin{bmatrix} t & 1 \\ 3 & 0 \end{bmatrix}$, $\underline{\underline{B}}(t) = \begin{bmatrix} t & \cos(t) \\ 1 & 0 \end{bmatrix}$

Can dif'te entry-wise.

$$\frac{d}{dt}\left(\underline{\underline{A}}(t)\right) = \begin{bmatrix} \frac{d}{dt} a_{11}(t) & \cdots & \frac{d}{dt} a_{1n}(t) \\ \frac{d}{dt} a_{m1}(t) & & \frac{d}{dt} a_{mn}(t) \end{bmatrix}$$

<u>Ex:</u> $\frac{d}{dt}\left(\underline{\underline{A}}(t)\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad \frac{d}{dt}\left(\underline{\underline{B}}(t)\right) = \begin{bmatrix} 1 & -\sin(t) \\ 0 & 0 \end{bmatrix}$

<u>Rules of dif'tion:</u>

1. If c constant scalar then
$$\frac{d}{dt}\left(c\underline{\underline{A}}(t)\right) = c\frac{d}{dt}\left(\underline{\underline{A}}(t)\right)$$

2. Product Rule:
$$\frac{d}{dt}\left(\underline{\underline{A}}(t)\,\underline{\underline{B}}(t)\right) = \left(\frac{d}{dt}\underline{\underline{A}}(t)\right)\underline{\underline{B}}(t) + \underline{\underline{A}}(t)\left(\frac{d}{dt}\underline{\underline{B}}(t)\right)$$

3.

Ex:

$$\frac{d}{dt}\left(\underline{\underline{A}}\,\underline{\underline{B}}\right) = \underset{\underline{\underline{A}}'}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}\underset{\underline{\underline{B}}}{\begin{bmatrix} t & \cos(t) \\ 1 & 0 \end{bmatrix}} + \underset{A}{\begin{bmatrix} t & 1 \\ 3 & 0 \end{bmatrix}}\underset{\underline{\underline{B}}'}{\begin{bmatrix} 1 & -\sin(t) \\ 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} t & \cos(t) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} t & -t\sin(t) \\ 3 & -3\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} 2t & \cos(t) - t\sin(t) \\ 3 & -3\sin(t) \end{bmatrix}.$$