Lesson 10 2/2/22

Ch 6: Non-linear systems

§6.1 Autonomous systems

Those are systems of the form

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$$

There is no dependence on t on the RHS

Ex:

$$\begin{cases} \frac{dx}{dt} = 2x+y \\ \frac{dy}{dt} = y \end{cases}$$

linear autonomous system

$$\begin{cases} \frac{dx}{dt} = \sin(y) \\ \frac{dy}{dt} = \cos(x) \end{cases}$$

autonomous, non-linear.

Non-ex:

$$\begin{cases} \frac{dx}{dt} = 2x+y+t \\ \frac{dy}{dt} = x+2y+\cos(t) \end{cases}$$

non-autonomous

linear, non-homog.

Existence & Uniqueness

If (x_0, y_0), t_0 are given and F, G are "nice" then there exists exactly one sol'n of

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$' = \frac{d}{dt}$

in an interval containing t_0 such that $x(t_0)=x_0$ and $y(t_0)=y_0$.

In general we can't write down explicit sol's for a general autonomous system.
There are some sol's that are easier to find.

<u>Observation:</u> If (x_0, y_0) is such that $F(x_0,y_0)=0$ <u>and</u> $G(x_0,y_0)=0$ then:

$$\begin{cases} x(t) = x_0 \\ y(t) = y_0 \end{cases} \quad (*)$$

is a sol'n to $\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$

<u>Check:</u> $x'(t) = x_0' = 0 = F(x_0,y_0) = F(x(t),y(t))$
similarly for $y(t)=y_0$, $y'(t) = G(x(t),y(t))$.

A point (x_0,y_0) such that $F(x_0,y_0)=0$ and $G(x_0,y_0)=0$ is called a Critical Point of the system.
If (x_0,y_0) is a CP then the constant sol'n $(x(t),y(t)) = (x_0,y_0)$ is an <u>equilibrium sol'n</u>.

<u>Ex 1:</u> $\begin{cases} x' = \sin(y) \\ y' = \cos(x) \end{cases}$

To find CP: $\sin(y)=0$ and $\cos(x)=0$

$\sin(y)=0 \Rightarrow y = k\pi, \quad k \in \mathbb{Z}$

$\cos(x)=0 \Rightarrow x = n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}$

So CP:

$$\left(n\pi + \frac{\pi}{2}, k\pi\right), \quad n, k \in \mathbb{Z}$$

Infinitely many.

<u>Ex 2:</u> $\begin{cases} x' = x + 2y \\ y' = 3x - y \end{cases}$

<u>CP</u>: $\begin{cases} x+2y=0 \\ 3x-y=0 \end{cases} \Rightarrow x=0 \;\&\; y=0$

So $(0,0)$ is the only CP.

<u>Fact</u>: For systems of form $\underline{x}' = \overset{2\times 2 \text{ matrix}}{\overbrace{\underline{\underline{A}}}}\,\underline{x}$ as in Ch 5, $(0,0)$ is <u>always</u> a CP.

<u>Ex 3:</u> $\begin{cases} x' = 2 - 4x - 15y \\ y' = 4 - x^2 \end{cases}$

Find CP.:

<u>Want</u>: $2-4x-15y=0$

$4-x^2=0 \Rightarrow x = \pm 2$

$x=2$: $\quad 2-8-15y=0 \Rightarrow y = -\frac{6}{15} = -\frac{2}{5}$

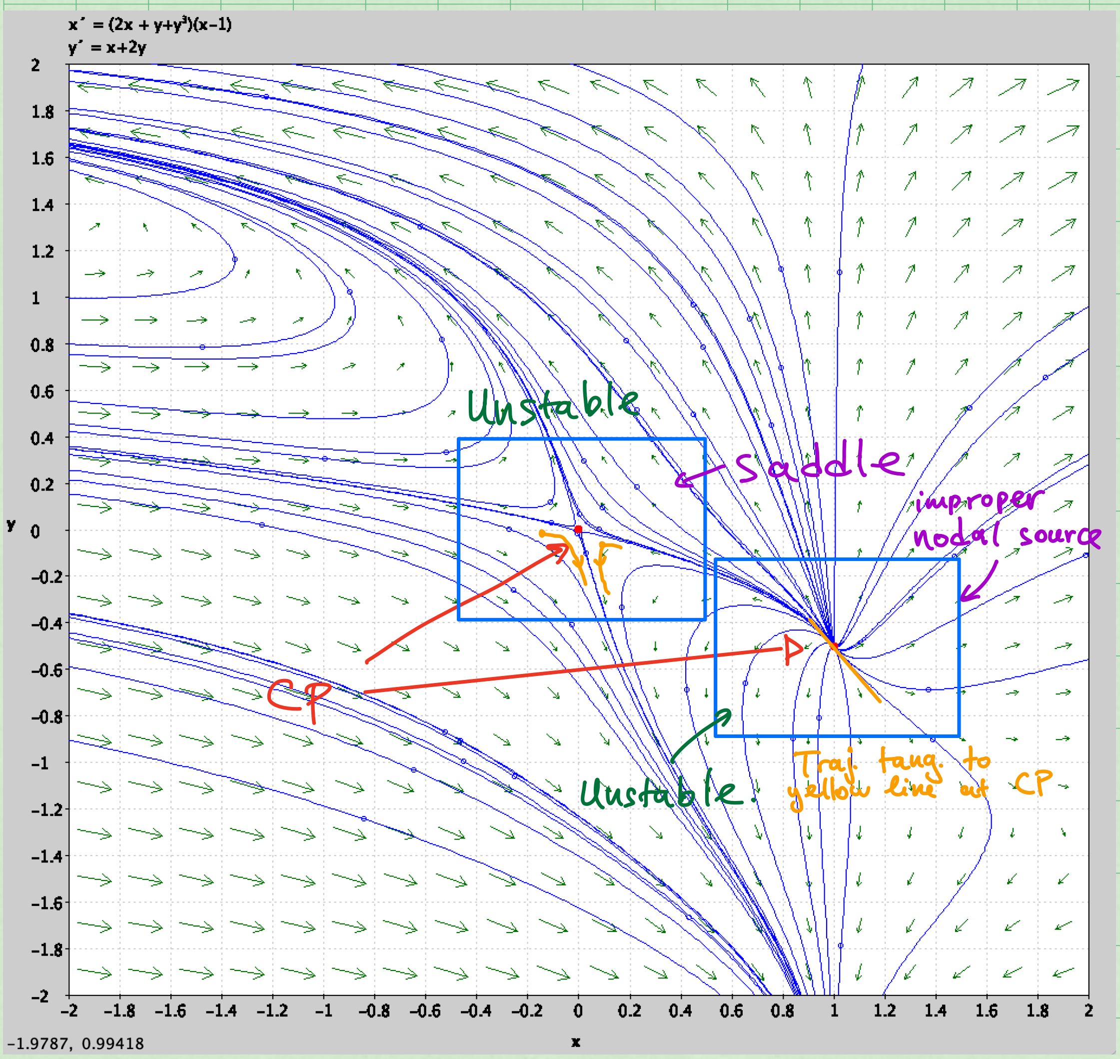
$x=-2$: $\quad 2+8-15y=0 \Rightarrow y = \frac{10}{15} = \frac{2}{3}$

So: CP: $\left(2, -\frac{2}{5}\right), \left(-2, \frac{2}{3}\right)$ //

We will use phase plane portraits to understand the behavior of autonomous systems:

$$\begin{cases} \dfrac{dx}{dt} = (x-1)(2x+y+y^3) \\ \dfrac{dy}{dt} = x+2y \end{cases}$$

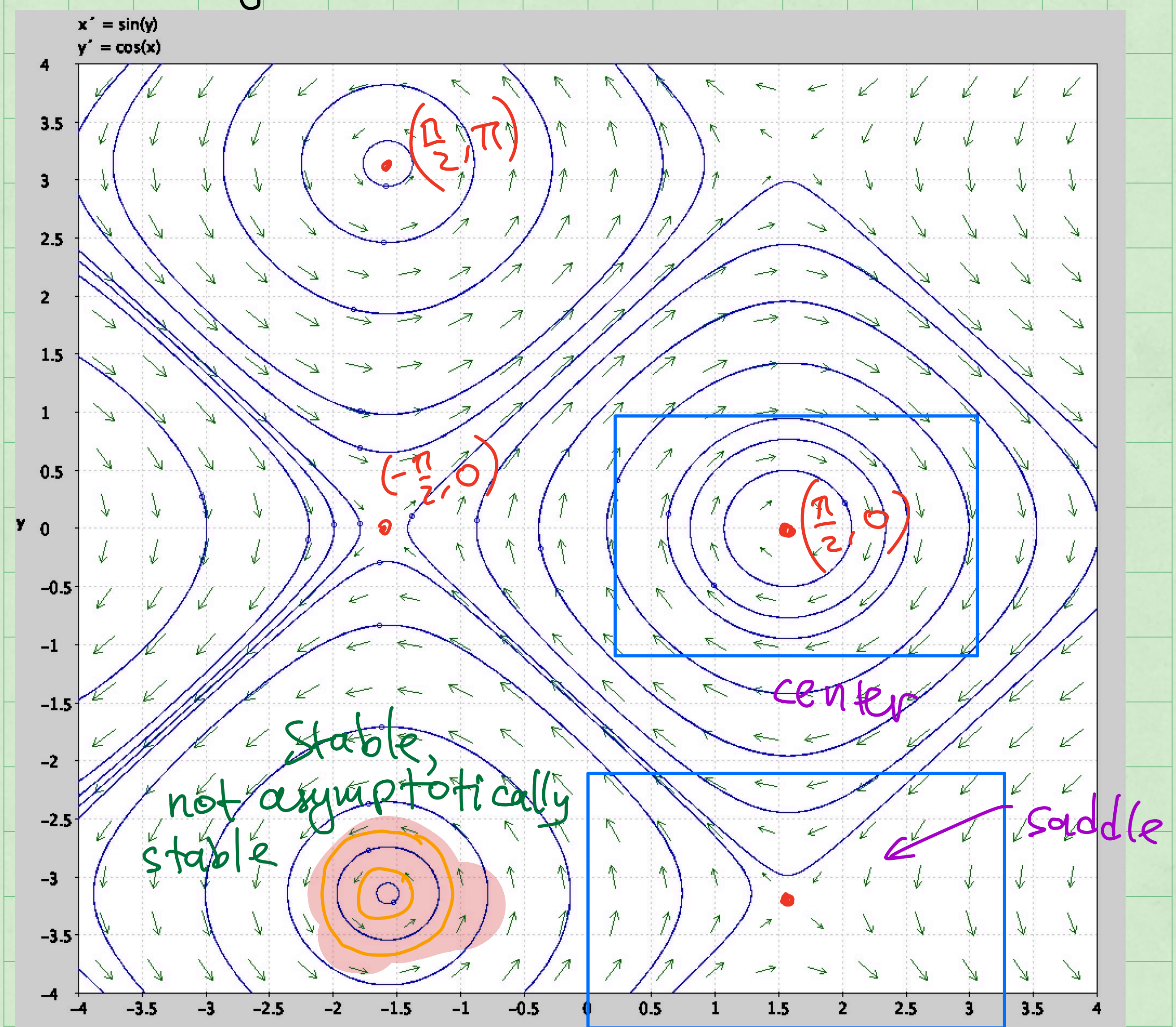
CP: $(0,0)$
$\left(1, -\frac{1}{2}\right)$



(We will classify the CP as saddles, nodes, centers...)

Ex: $x' = \sin(y)$ $\quad$ CP: $\left(k\pi + \frac{\pi}{2}, n\pi\right)$

$y' = \cos(x)$ $\quad$ $k, n \in \mathbb{Z}$



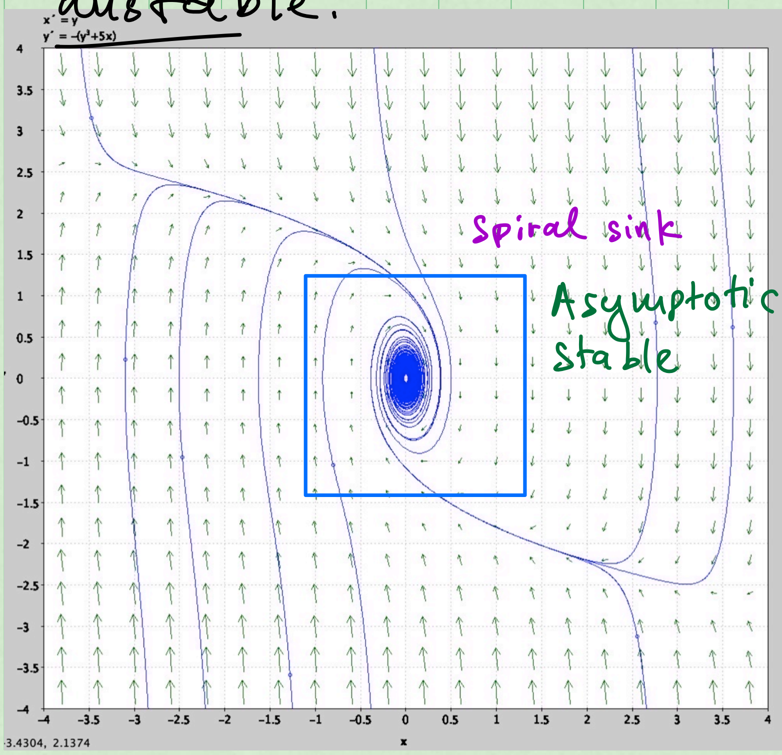
Stable/ unstable CP

Stable: Every trajectory which starts sufficiently close to the CP stays close to it for all time (but it doesn't have to converge to the CP).

If any trajectory which starts sufficiently close to the CP converges to it as $t \to \infty$ the CP is called asymptotically stable

Asymptotically stable $\Rightarrow$ stable
$\not\Leftarrow$

A C.P. which is not stable is called unstable.



Generally: nodal sinks, spiral sinks are asymptotically stable

centers: stable, not asymptotically stable

nodal sources, spiral sources, saddles: unstable