Lesson 11

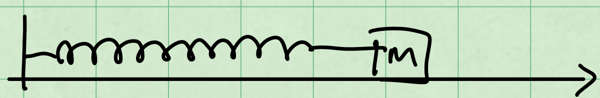
04/02/2022

Last time: autonomous systems

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

C.P. : (x_0, y_0) such that $F(x_0,y_0) = G(x_0,y_0) = 0$

Ex: Mechanical spring-mass system



$$\underset{\uparrow \text{ mass}}{m}\, x'' + \underbrace{c\,(x')^3}_{\text{damping}} + \underset{\uparrow \text{ spring constant}}{k}\, x = 0$$

Can be studied as an autonomous (nonlinear) system

set $y = x'$ ← velocity

$$y' = x'' = -\frac{1}{m}\left(c y^3 + k x\right)$$

So:

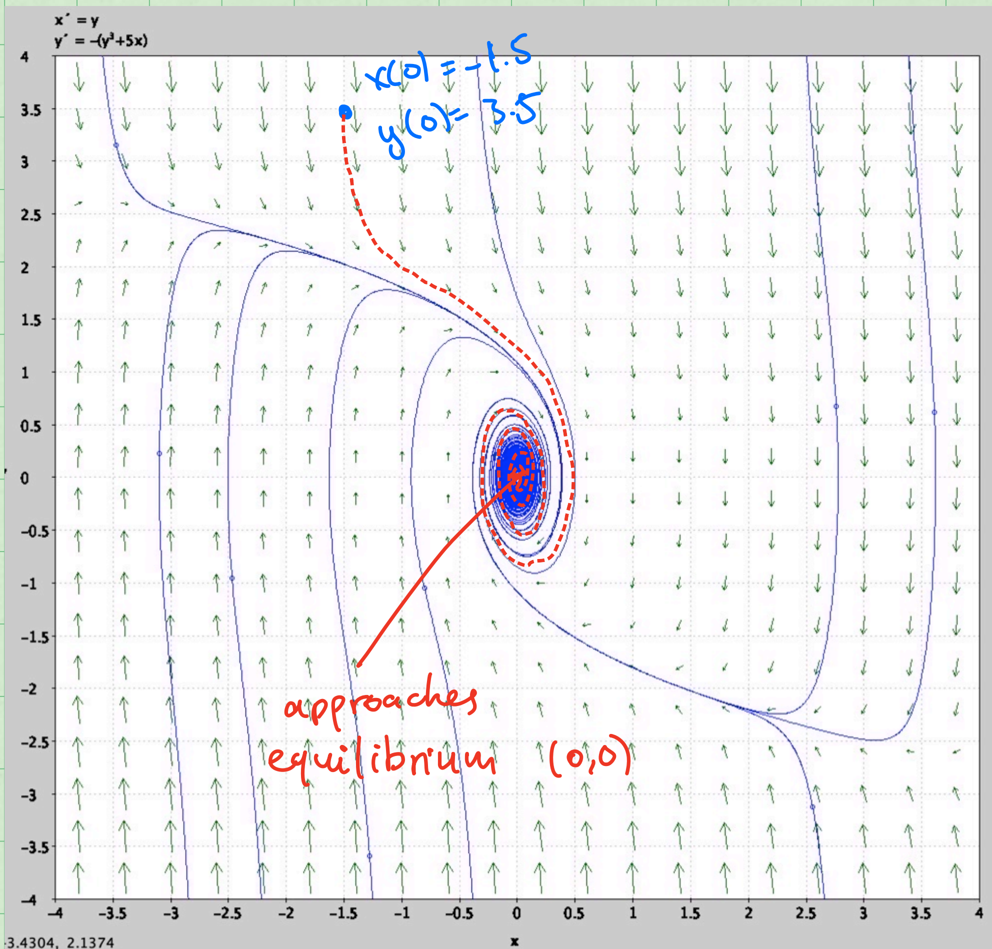
$$\begin{cases} x' = y \\ y' = -\frac{1}{m}\left(c y^3 + k x\right) \end{cases}$$

Autonomous (no t on RHS), nonlinear.

C.P. $\quad y = 0, \quad -\frac{1}{m}(c \cdot 0 + kx) = 0$

$\Rightarrow x = 0$

Only $(0,0)$



$m = 1$

$c = 1$

$k = 5$

y: velocity

x : displacement

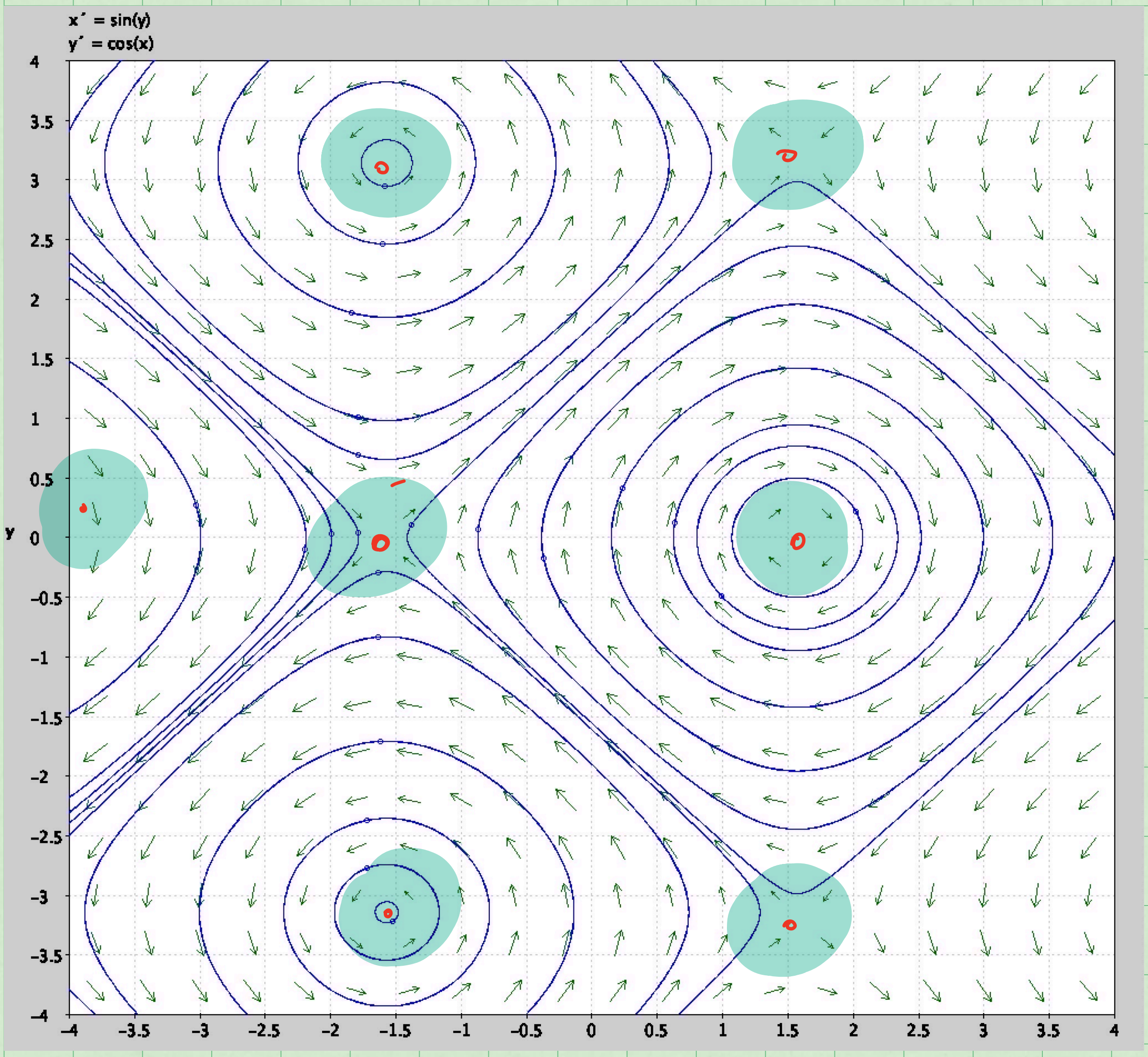
6.2 Analyzing non-linear systems near C.P.

Def'n: A C.P. is called isolated if there is a neighborhood of it which contains no other C.P.

Ex:
$$\begin{cases} x' = \sin(y) \\ y' = \cos(x) \end{cases}$$

C.P. : $\left(k\pi + \frac{\pi}{2}, n\pi\right)$, $k, n \in \mathbb{Z}$

Infinitely many, isolated



Each green are contains only one CP

Fact: if there are finitely many CP

they are isolated.

Non-example: $x' = x$
$y' = x$

CP: $(x,y) = (0,y)$ for any y

Every pt on y-axis is a CP, can't "separate them". //

Recall: Taylor's Theorem

If $f(x,y)$ is nice, (x_0,y_0) is a given point, then:

$$f(x_0+u,\ y_0+v) = \underbrace{f(x_0,y_0)}_{\text{constant in } u,v} + \underbrace{\underbrace{\partial_x f\big|_{(x_0,y_0)}}_{\text{value of } \partial_x f \text{ at } (x_0,y_0)} u + \partial_y f\big|_{(x_0,y_0)} v}_{\text{linear in } (u,v)} + \underbrace{r(u,v)}$$

u,v small

$x = x_0 + u,\ y = y_0 + v$

measures how "far" we are from being linear near (x_0,y_0)

where $r(u,v)$ is an error term w/

$$\underbrace{\lim_{(u,v)\to(0,0)} \frac{r(u,v)}{\sqrt{u^2+v^2}} = 0}_{\text{error small relative to } |(u,v)|}.$$

← this is true for nice functions

Ex: $f(x,y) = e^{-x^2-y^2}$

$(x_0,y_0) = (1,0)$

Write Taylor's theorem:

$f(1,0) = e^{-1}$

$\partial_x f(x,y) = -2x\, e^{-x^2-y^2} \Rightarrow \partial_x f(1,0) = -2e^{-1}$

$\partial_y f(x,y) = -2y\, e^{-x^2-y^2} \Rightarrow \partial_y f(1,0) = 0$

So:

$$f(1+u, v) = e^{-1} + (-2e^{-1})u + 0\cdot v + \underbrace{r(u,v)}_{\text{small}}$$

so f is approximated near $(1,0)$ by

$$f(1+u,v) \approx e^{-1} - 2e^{-1}u$$

//

Given autonomous system:

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y) \end{cases} \quad (*)$$

Let (x_0,y_0) be a C.P. Apply Taylor's thm at (x_0,y_0) for f and g.

Write: $\quad x = x_0 + u, \quad y = y_0 + v$

$$\Rightarrow \frac{dx}{dt} = \frac{du}{dt}, \quad \frac{dy}{dt} = \frac{dv}{dt}$$

$$(*)\begin{cases} \dfrac{du}{dt} = \overbrace{\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} u + \left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)} v + r(u,v)}^{\text{Taylor for } f \text{ w.r.t. } (x_0,y_0)} \\ \dfrac{dv}{dt} = \underbrace{\left.\dfrac{\partial g}{\partial x}\right|_{(x_0,y_0)} u + \left.\dfrac{\partial g}{\partial y}\right|_{(x_0,y_0)} v + s(u,v)}_{\text{Taylor for } g \text{ w.r.t. } (x_0,y_0)} \end{cases}$$

(x_0,y_0): CP so $f(x_0,y_0) = g(x_0,y_0) = 0$.

If u, v small, $r(u,v)$, $s(u,v)$ insignificant, truncate $(*)$:

$$\begin{cases} \dfrac{du}{dt} = \partial_x f|_{(x_0,y_0)}\, u + \partial_y f|_{(x_0,y_0)}\, v \\ \dfrac{dv}{dt} = \partial_x g|_{(x_0,y_0)}\, u + \partial_y g|_{(x_0,y_0)}\, v \end{cases}$$

to obtain a linear system in terms of (u,v).

Called the linearized system associated to $(*)$ at (x_0,y_0)

Matrix of linearized system: Jacobian

$$\underline{\underline{J}}(x_0,y_0) = \begin{bmatrix} \partial_x f|_{(x_0,y_0)} & \partial_y f|_{(x_0,y_0)} \\ \partial_x g|_{(x_0,y_0)} & \partial_y g|_{(x_0,y_0)} \end{bmatrix}$$

so $\underline{u}' = \underline{\underline{J}}\,\underline{u}$

<u>Ex:</u>

$$\begin{cases} \frac{dx}{dt} = e^{x+y} - 1 \\ \frac{dy}{dt} = x^3 + y \end{cases}$$

non-linear autonomous

<u>Find CP:</u> $e^{x+y} - 1 = 0 \Rightarrow x+y=0$

$x^3 + y = 0 \Rightarrow x^3 - x = 0 \Rightarrow x(x^2-1)=0$

$\Rightarrow x=0,\ x=1,\ x=-1$

<u>CP:</u> $(0,0)$, $(1,-1)$, $(-1,1)$

<u>Note:</u> finitely many $\Rightarrow$ isolated

Compute linearization at $(0,0)$

$$\underline{\underline{J}}(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ 3x^2 & 1 \end{bmatrix}$$

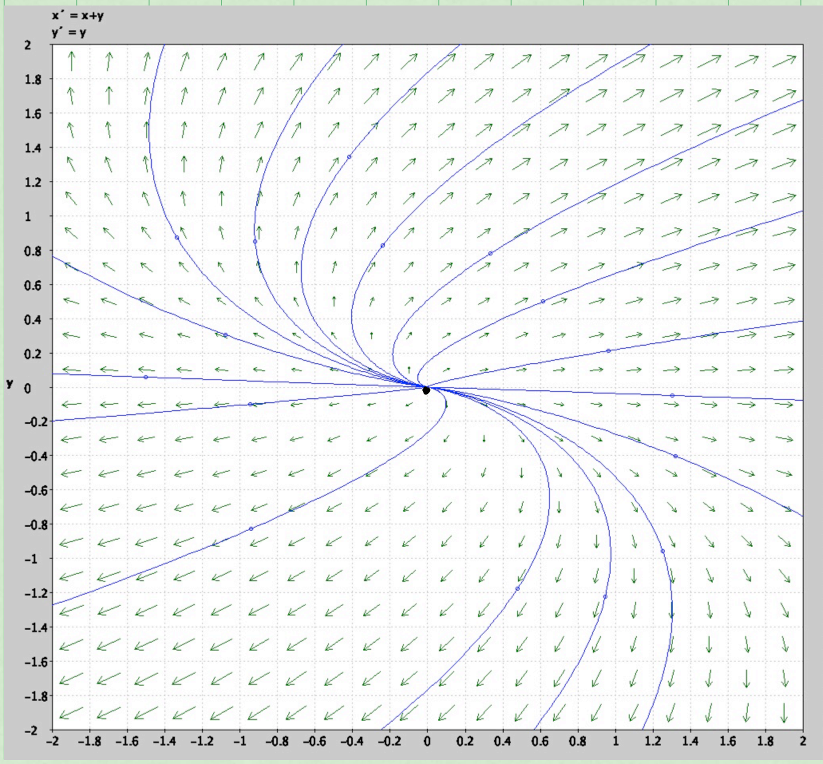
$$\begin{array}{cc} 3x^2 = \partial_x(x^3+y) & 1 = \partial_y(x^3+y) \end{array}$$

So: at $(0,0)$

$$\underline{\underline{J}}(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

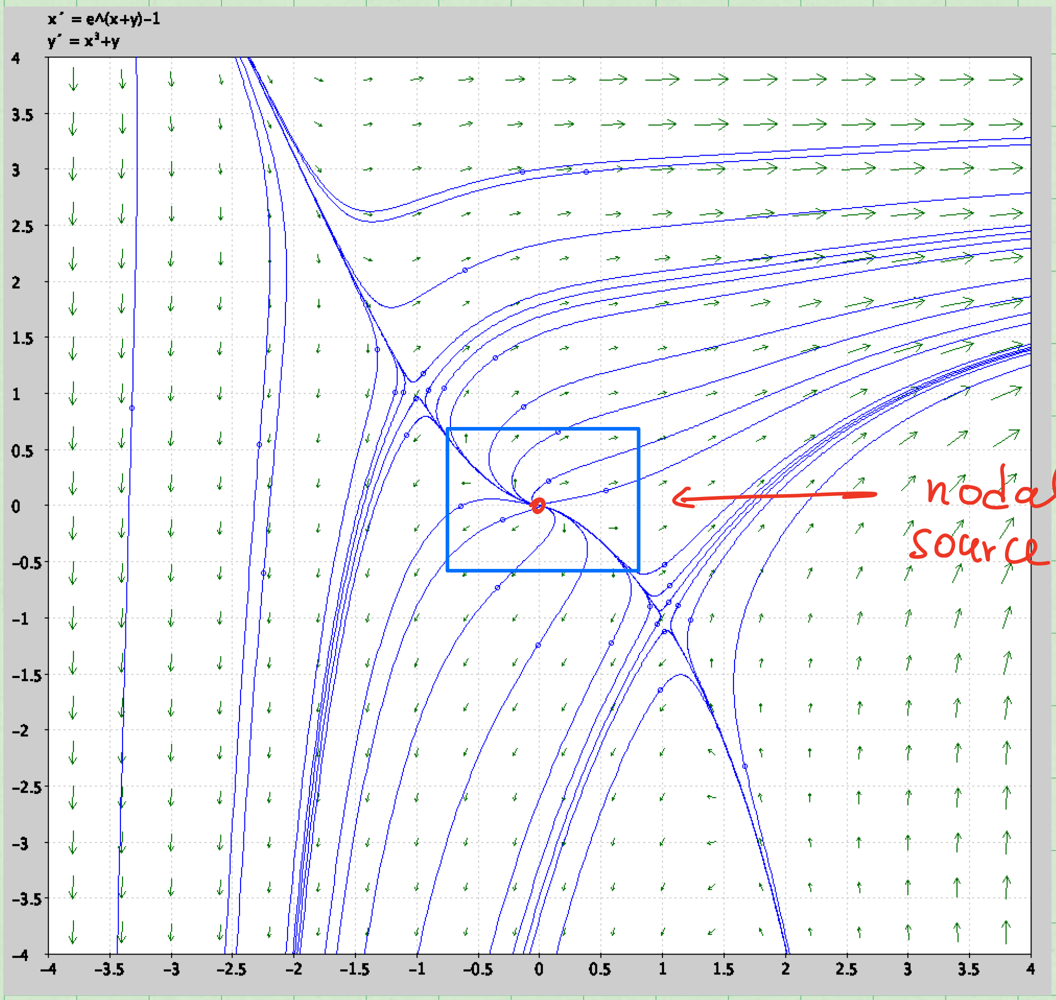
and linearization at $(0,0)$ is

$$\begin{cases} u' = u + v \\ v' = v \end{cases}$$



phase plane portrait of linearized system

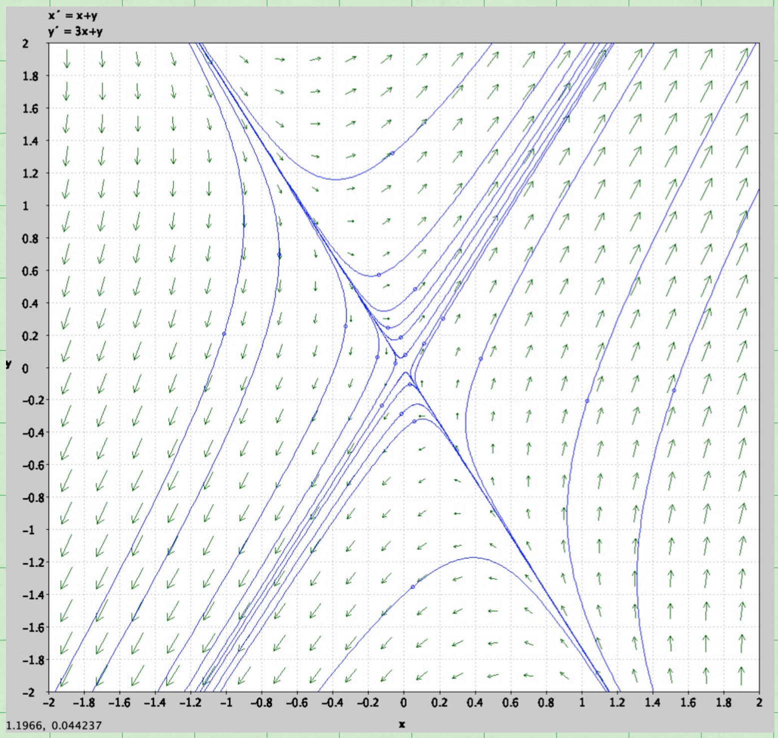
nodal source.

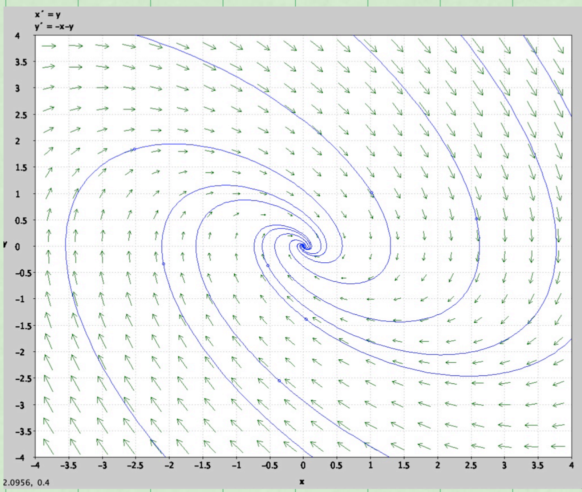


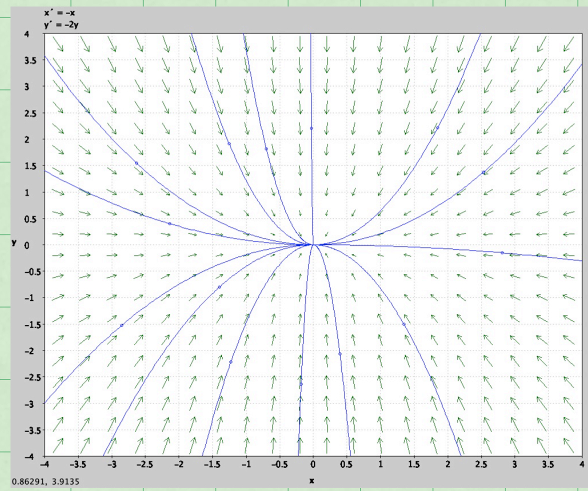
nonlinear system (original)

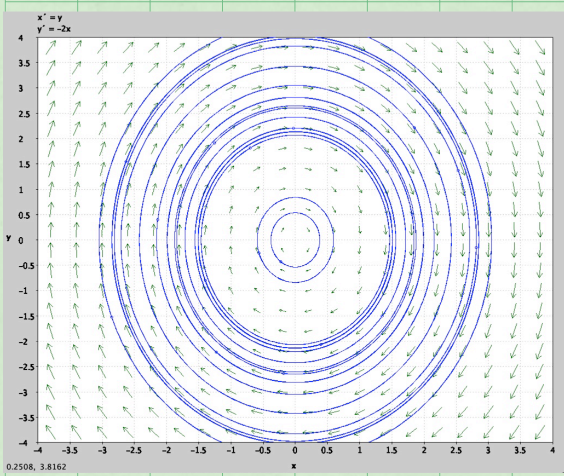
nodal source

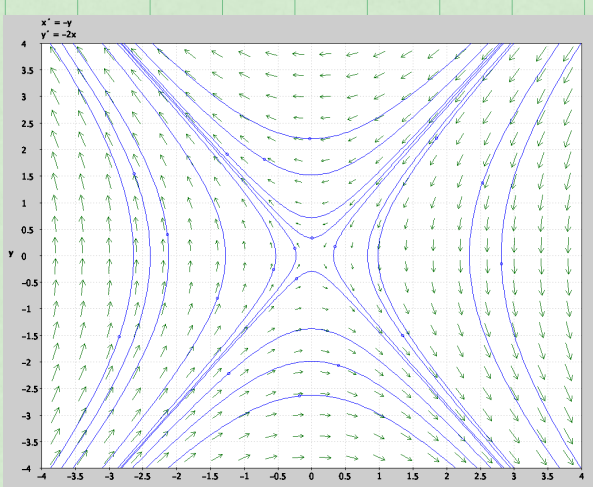
<u>Exercise:</u> compute linearizations at $(1,-1)$, $(-1,1)$ //.

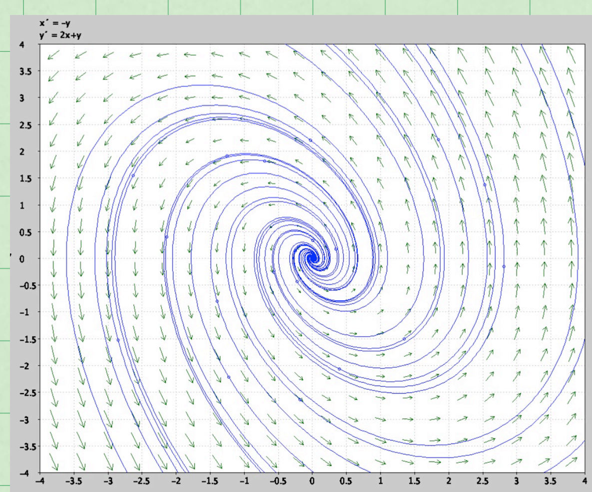

x' = x+y
y' = 3x+y
x
y
1.1966, 0.044237


x' = y
y' = −x−y
2.0956, 0.4


x' = −x
y' = −2y
0.86291, 3.9135


x' = y
y' = −2x
0.2508, 3.8162


x' = −y
y' = −2x
0.021254, −2.8541


x' = −y
y' = 2x+y
2.6397, 0.36757