Lesson 12

02/07/22

Last time:

$$\begin{cases} \dfrac{dx}{dt} = e^{x+y} - 1 \\[2ex] \dfrac{dy}{dt} = x^3 + y \end{cases}$$

found C.P.: $(0,0)$, $(1,-1)$, $(-1,1)$

Computed linearization at $(0,0)$

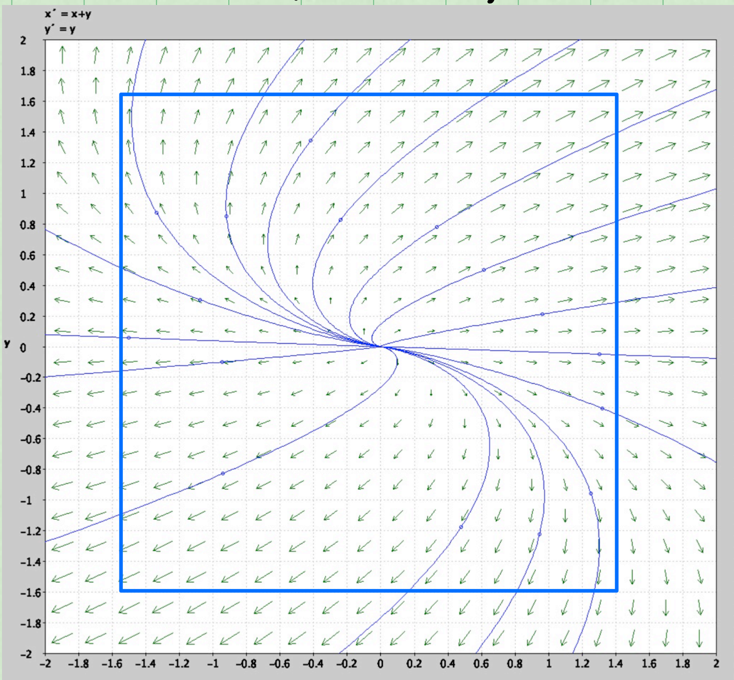
$$J(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ 3x^2 & 1 \end{bmatrix}$$

At $(0,0)$: linearization

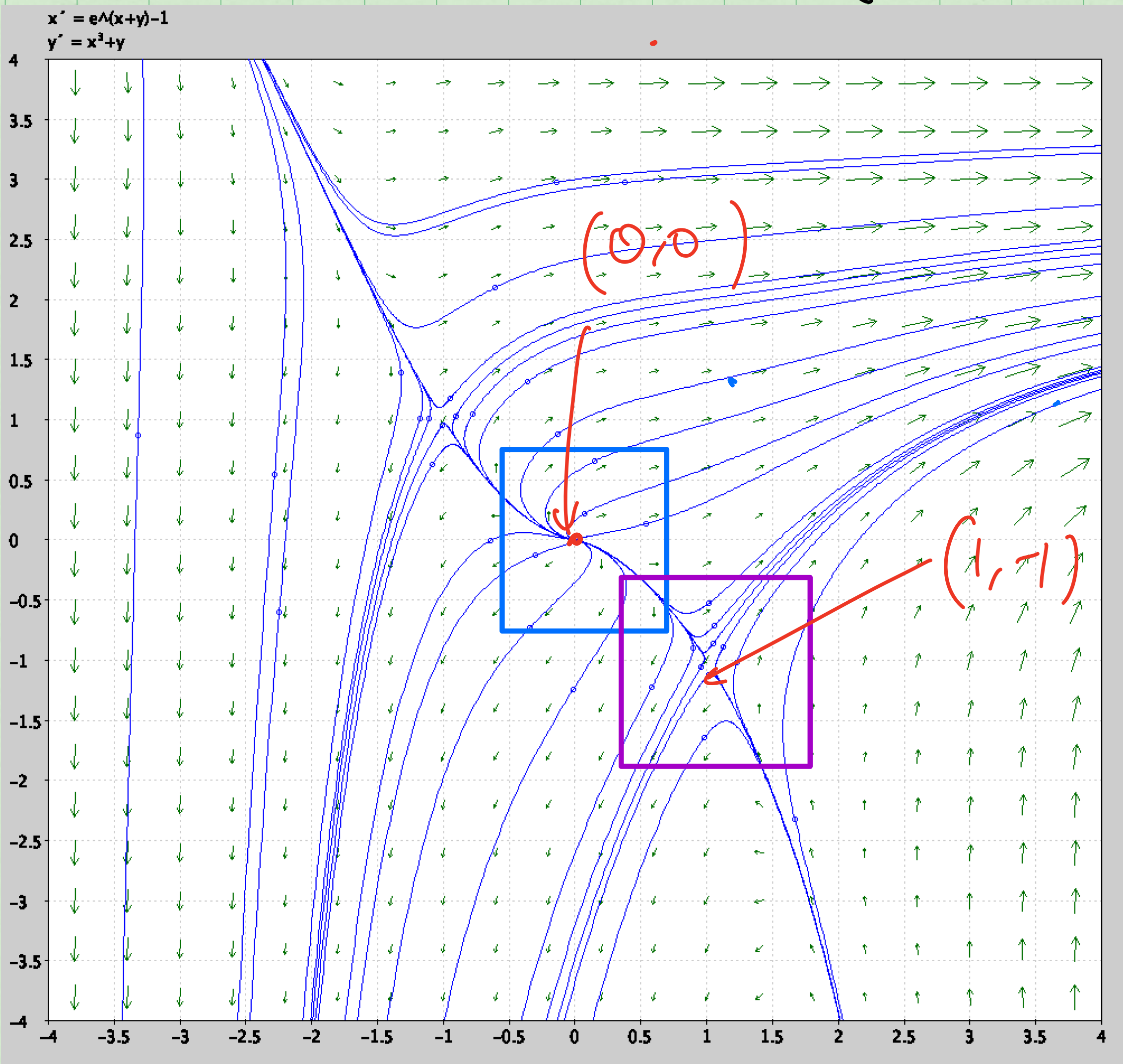
$$\underline{u}' = \underline{J}(0,0)\,\underline{u} \Rightarrow \underline{u}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underline{u}$$

E-values of linearized system: $\lambda = 1$ repeated

Phase plane portrait: nodal source:



Compare to nonlinear system:



Rmk: behavior of linearized system depends on where we are linearizing.

Ex: (1,-1) also a CP in our example.

$$J(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ 3x^2 & 1 \end{bmatrix}$$

Linearization at $(1,-1)$

$$\underline{u}' = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \underline{y}'$$

e-values:

$$(1-\lambda)^2 - 3 = 0 \Rightarrow \lambda = \underbrace{1 \pm \sqrt{3}}$$

2 real e-values of opposite signs.

Phase Plane Portrait for linearized system: saddle.

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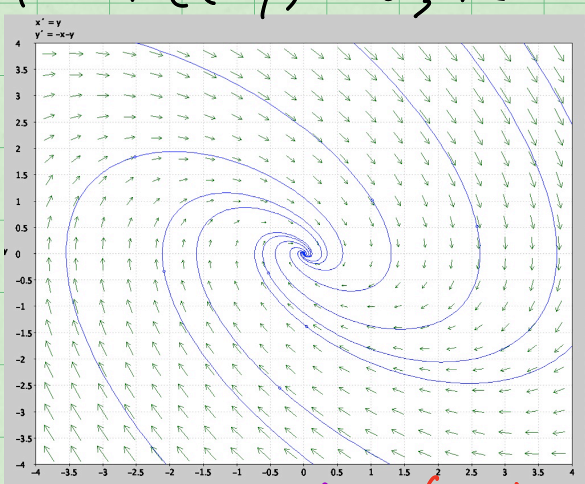
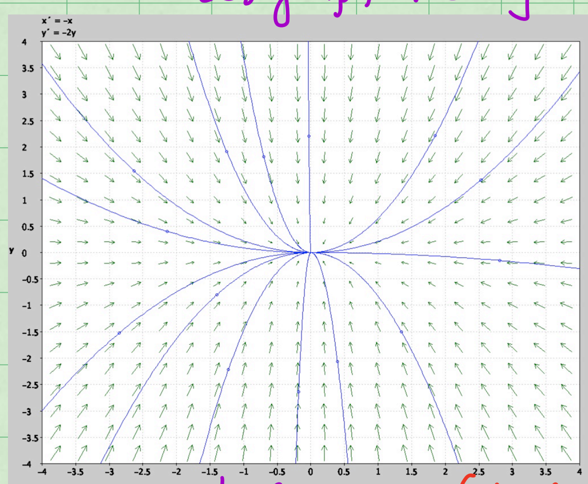
Recall: how eigenvalues determine stability properties of phase plane portraits.

1. If $Re(\lambda_1) < 0$, $Re(\lambda_2) < 0$: asymptotically stable



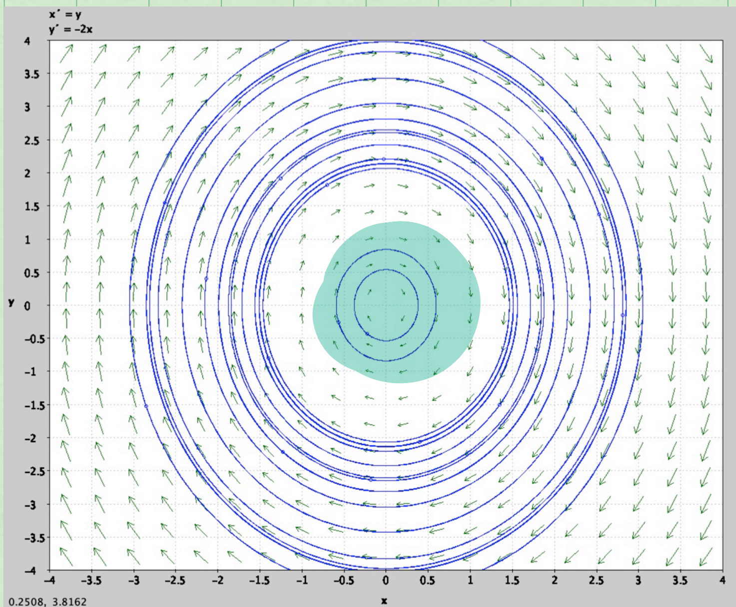
spiral sink ($Im\ \lambda_j \neq 0$)



nodal sink ($Im(\lambda_j) = 0$)

2. If $Re(\lambda_1) = Re(\lambda_2) = 0$, $Im(\lambda_j) \neq 0$

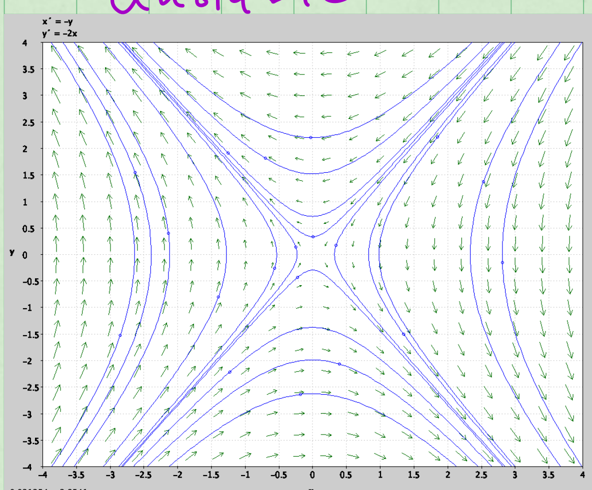
$$\lambda_{1,2} = \pm ai$$



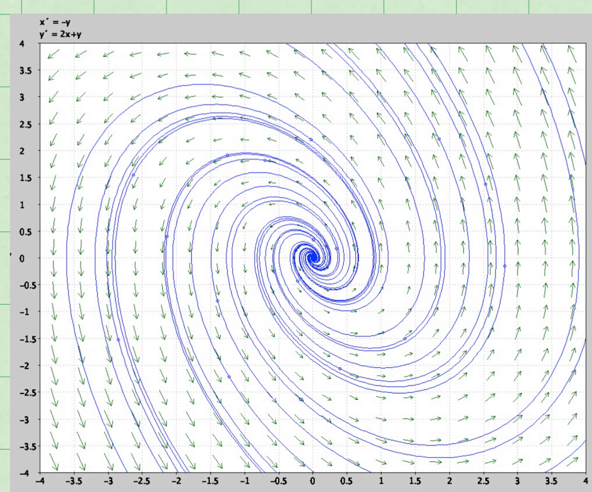
Stable center, not asymptotically stable.

3. $Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$ or $Re(\lambda_1) > 0$, $Re(\lambda_2) < 0$

Unstable



Saddle



Spiral source

Idea: A nonlinear system will behave near one of its critical pts similarly to a perturbation of the linearized system at the C.P.

Preparation: what happens to evalues of linear system when it is perturbed.

Ex: $\underline{\underline{x}}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \underline{\underline{x}}$

E-values: $\lambda = -1$ repeated

phase plane portrait:

(proper) nodal sink, as. stable

Perturb:

a) $\underline{\underline{x}}' = \begin{bmatrix} -1+0.1 & 0 \\ 0 & -1-0.1 \end{bmatrix} \underline{\underline{x}}'$

E-values: -0.9, -1.1, so not repeated, improper nodal sink, asymptotically stable.

b) $\underline{\underline{x}}' = \begin{bmatrix} -1 & 0.1 \\ -0.1 & -1 \end{bmatrix} \underline{\underline{x}}$

$(-1-\lambda)^2 + 0.1^2 = 0 \Rightarrow \quad \lambda = -1 \pm i\,0.1$

E-values: not repeated, not real,
real pt still negative

PPP: as. stable spiral sink.

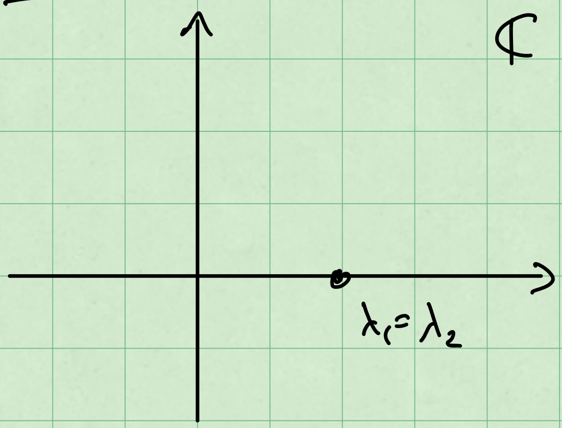
In this example: $Re(\lambda) < 0$ was preserved
under small perturbations
(so as. stability too)
<u>but</u> $Im(\lambda_{1,2}) = 0$ or
$\lambda_1 = \lambda_2$ were <u>not</u> preserved.

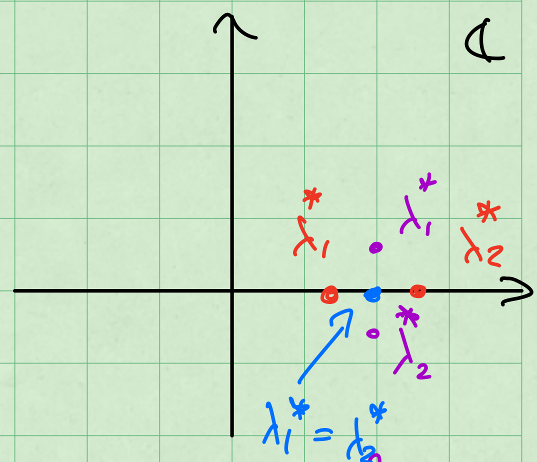
In principle: inequality preserved under
perturbations, equality might
not be.

How eigenvalues behave under perturbations, in pictures. All systems have real coefficients.

Original | Perturbed

Case 1 $\lambda_1 = \lambda_2 > 0$



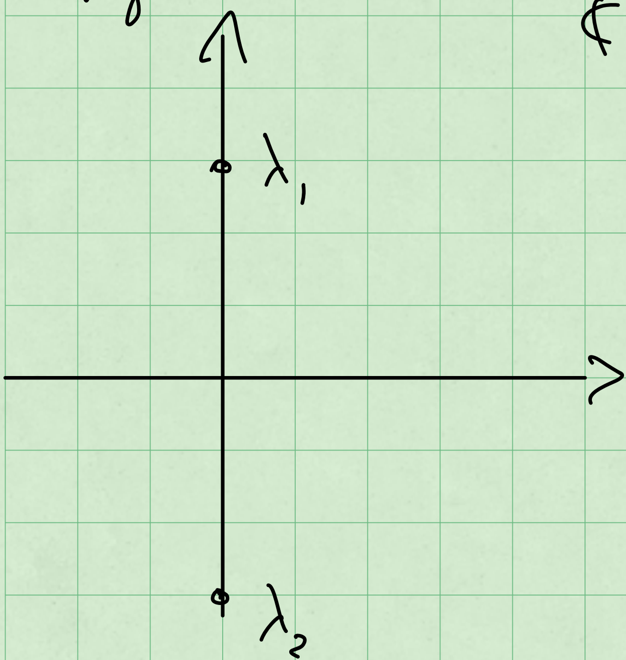


- Complex Conjugate e-values, real pt > 0
  unstable spiral source.
- Real, not repeated, positive
  unstable nodal source (improper)
- Real, repeated, positive
  unstable nodal source. (proper or improper)

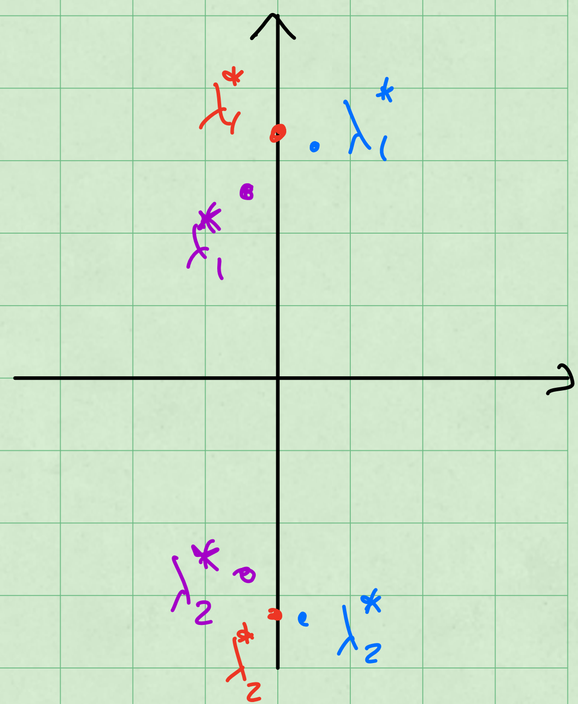
If $\lambda_1 = \lambda_2 < 0$ similar, as. stable.

Case 2: Purely imaginary

Original



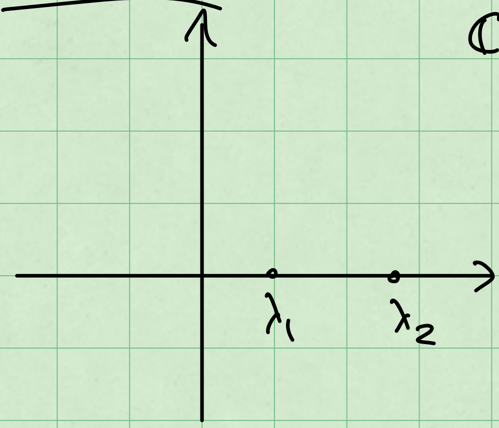
&

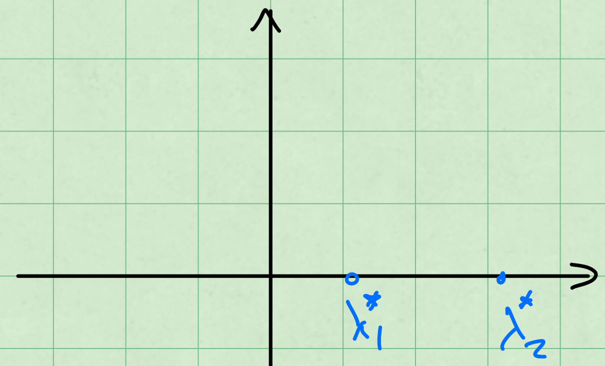


- λ_1^*, λ_2^* positive real pt, unstable spiral source
- λ_1^*, λ_2^* negative real pt, as. stable spiral sink.
- λ_1^*, λ_2^* purely imaginary, stable center (not as. stable)

If $\lambda_{1,2}$ purely imaginary can't predict stability of perturbed system.

Case 3: distinct real e-values $\neq 0$





☞ still real distinct e-values, of same signs $\Rightarrow$ stability properties of perturbed system same as unperturbed.

For non-linear systems:

Def'n: Let
$$x' = f(x,y)$$
$$y' = g(x,y) \quad (*)$$
be an autonomous system, f, g nice. Let (x_0, y_0) be a C.P. We say that $(*)$ is an Almost Linear System (ALS) for (x_0, y_0) if (x_0, y_0) is an isolated CP and 0 is not an eigenvalue of linearized system at (x_0, y_0).

Ex: $\dfrac{dx}{dt} = e^{x+y} - 1$, $\dfrac{dy}{dt} = x^3 + y \quad (*)$

Found: CP finitely many $(0,0)$, $(1,-1)$, $(-1,1)$

$\Rightarrow$ isolated.

E-values of linearization at $(0,0)$

$\lambda = 1$ repeated

so $(*)$ is ALS for $(0,0)$.

Check: ALS for $(1,-1)$, $(-1,1)$ as well.

Non-ex:
$$\begin{cases} \dfrac{dx}{dt} = x - y^2 \\ \dfrac{dy}{dt} = x + y^2 \end{cases}$$

$(0,0)$ CP

linearized system has e-value 0.

Thm: Given an ALS at (x_0, y_0), let λ_1, λ_2 be the e-values of linearized system at (x_0, y_0).

1. If λ_1, λ_2 real & equal, CP (x_0, y_0) is either node or spiral
   - $\rightarrow$ as. stable if $\lambda_1, \lambda_2 < 0$
   - $\rightarrow$ unstable if $\lambda_1, \lambda_2 > 0$
2. If λ_1, λ_2 purely imaginary, CP either center or spiral, stable or unstable.

(3) In all other cases: type and stability of CP same as for linearized system.

Idea: ALS behaves like a perturbation of linearized system.