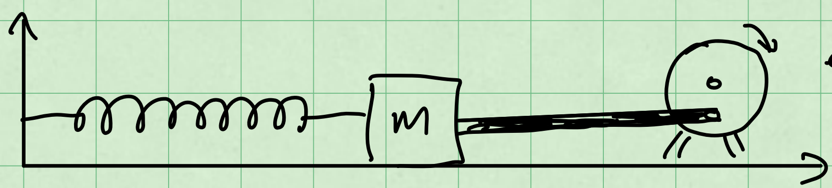
Lesson 16

02/16/2022

§ 7.1

Plan: introduce the Laplace transform.



external force, depending on time

displacement from equil (linear spring):

non-homogeneous linear const. coefficient 2nd ODE $\longrightarrow$ $$m x'' + \underbrace{c x'}_{\text{damping term}} + \underbrace{k}_{\text{spring const.}} x = \underbrace{f(t)}_{\text{external force (source)}} \quad (*)$$

In 262/266: found general sol'n for homogeneous version: $m x'' + c x' + k x = 0$ w/ method of characteristics, found a particular sol'n for $(*)$, sol'n for $(*)$

$$x = x_{\text{gen/hom.}} + x_{\text{part.}}$$

Now: New tool for solving $(*)$ in a more direct way, can deal w/ more complicated sources

Laplace Transform:

Given $f(t)$ defined for $t \geq 0$, its Laplace transform is a function of a new variable $s \in \mathbb{R}$

capital letter corr. to input function.

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{M\to\infty} \int_0^M e^{-st} f(t)\,dt$$

L for Laplace

for the $s \in \mathbb{R}$ for which the integral converges.

$$\text{functions of } t \xrightarrow{\text{Laplace}} \text{functions of } s.$$

Ex 1: $f(t) = e^{at}$, a real, $t \geq 0$
$\mathcal{L}\{f(t)\}(s) = ?$, for what s does it make sense?

Ex 2: $f(t) = 1$ same

1: $\mathcal{L}\{f(t)\} = \lim\limits_{M\to\infty} \int_0^M e^{-st} e^{at}\,dt$

$= \lim\limits_{M\to\infty} \int_0^M e^{(a-s)t}\,dt$

$s \neq a$

$= \lim\limits_{M\to\infty} \left[\dfrac{e^{(a-s)t}}{a-s} \right]_0^M = \lim\limits_{M\to\infty} \left(\dfrac{e^{(a-s)M}}{a-s} - \dfrac{1}{a-s} \right)$

$s > a$

$$= \frac{1}{s-a}, \quad s > a.$$

2. Special case of 1 for $a = 0$

$$\mathcal{L}\{1\}(s) = \frac{1}{s}, \quad s > 0.$$

Rmk 1: $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ also for

$a \in \mathbb{C}, \quad s > \mathrm{Re}(a)$

Rmk 2: Laplace makes sense for (nice) functions which satisfy

$$|f(t)| \leq M e^{ct}$$

for large t, for some M, c.
Then $\mathcal{L}\{f(t)\}(s)$ is defined for $s > c$.

Ex: $f(t) = 5e^{2t}\sin(t)$

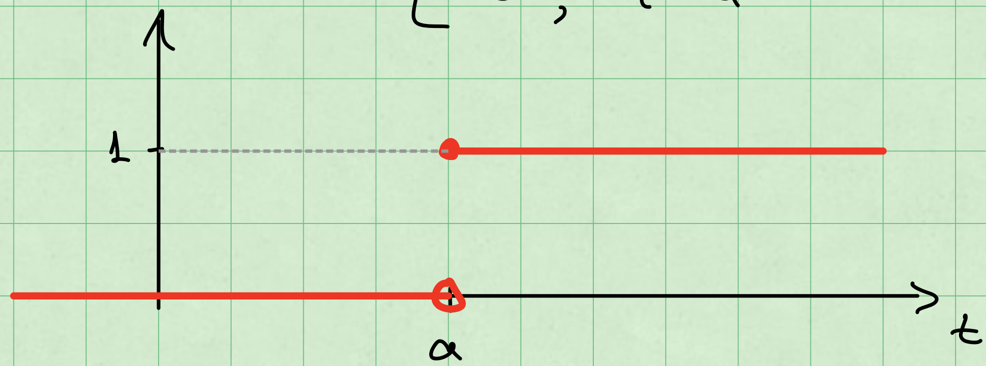
$$|f(t)| = 5e^{2t}|\sin(t)| \leq 5e^{2t}$$

$\Rightarrow$ $\mathcal{L}\{f(t)\}$ makes sense for $s > 2$.

Non-example: $f(t) = e^{t^2}$ (bad)

Ex 3: Step functions

$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \qquad a \in \mathbb{R}$$



If $a=0$, $u(t) := u_0(t)$

(also written as $H(t)$ after Heaviside)

With this notation: $u_a(t) = u(t-a)$

(check!)

$$\mathcal{L}\{u_a(t)\}(s) = \int_0^\infty e^{-st} u_a(t)\, dt \qquad (a > 0)$$

$$= \int_0^a e^{-st} \underbrace{u_a(t)}_{=0} dt + \int_a^\infty e^{-st} \underbrace{u_a(t)}_{=1} dt$$

$$= \lim_{M \to \infty} \int_a^M e^{-st}\, dt$$

$$\overset{s\neq 0}{=} \lim_{M\to\infty} \left[\frac{e^{-st}}{-s} \right]_a^M$$

$$= \lim_{M\to\infty} \left(\frac{e^{-sM}}{-s} - \frac{e^{-sa}}{-s} \right)$$

$$\overset{s>0}{=} \frac{e^{-as}}{s}, \quad s > 0$$ //

Read: Gamma function p. 439.

Laplace tr. is linear:

If a, b are const:

$$\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$$

Means we can often use tables

Ex: $\mathcal{L}\{3t^4 + 5\cosh(3t)\}$

$$= 3\mathcal{L}\{t^4\} + 5\mathcal{L}\{\cosh(3t)\}$$

$$= 3\frac{4!}{s^5} + 5\frac{s}{s^2-9} \quad //.$$

Next time: → Inverse Laplace transform

$F(s) \longrightarrow f(t)$

→ Use Laplace to solve ODE //.