Lesson 17 (7.1 - 7.2) 02/18/2022

Recall:

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

If $F(s) = \mathcal{L}\{f(t)\}$ then $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is the inverse Laplace transform of $F(s)$

Process for finding $\mathcal{L}^{-1}$: break down $F(s)$ into simple functions for which $\mathcal{L}^{-1}$ can be found using tables.

Ex 1: $$\mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= t + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= t + \frac{1}{2}\sinh(2t)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sinh(at)}$$

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Ex 2: $$F(s) = \frac{1}{s(s^2+4s+3)}$$

We'll use partial fractions.

1. Factor denominator

$$s^2+4s+3 = (s+3)(s+1)$$

2. Write:

$$\frac{1}{s(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}$$

$$1 = A(s+3)(s+1) + B(s+1)s + C(s+3)s$$

For A: set $s=0 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$

For B: set $s=-3 \Rightarrow 1 = 6B \Rightarrow B = \frac{1}{6}$

For C: set $s=-1 \Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$

So: $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

$$-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$\overset{\text{table}}{=} \frac{1}{3}\cdot 1 + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}$ //

$\cancel{e^{t^2}}$

Property: If f nice $|f(t)| \leq Me^{ct}$, (const.)

then:

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ (depends on s)

Differentiation $\xrightarrow{\mathcal{L}}$ Multiplication by s.

PP: $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$

$$\overset{\text{IBP}}{=} e^{-st} f(t) \Big|_0^\infty - \int_0^\infty \left(e^{-st}\right)' f(t)\, dt$$

$$= \underbrace{\lim_{M \to \infty} \left(e^{-sM} f(M)\right)}_{\substack{\text{if } |f(t)| \le M e^{ct} \\ \to 0 \text{ for large } s}} - f(0) + s \underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{\mathcal{L}\{f(t)\}}$$

$$= s\,\mathcal{L}\{f(t)\} - f(0). \quad //$$

Ex 3: $\begin{cases} 4x' + 3x = 1 & (1) \\ x(0) = 0 & (2) \end{cases}$

Use property to solve IVP like (1)-(2)

Apply $\mathcal{L}$ on both sides of (1):

$$4\,\mathcal{L}\{x'\} + 3\,\mathcal{L}\{x\} = \mathcal{L}\{1\} \quad (*)$$

write: $X(s) = \mathcal{L}\{x(t)\}$

$$(*) \overset{\text{property}}{\Longrightarrow} 4\left(s\,\mathcal{L}\{x(t)\} - \underbrace{x(0)}_{0 \text{ by } (2)}\right) + 3\,\mathcal{L}\{x(t)\} = \frac{1}{s}$$

$$\Rightarrow 4sX(s) + 3X(s) = \frac{1}{s}$$

$$\Rightarrow \quad X(s) = \frac{1}{s(4s+3)}$$

Take $\mathcal{L}^{-1}$: use Partial Fractions

$$\frac{1}{s(4s+3)} = \frac{1}{4s(s+\frac{3}{4})} = \frac{A}{s} + \frac{B}{s+\frac{3}{4}}$$

$$\Rightarrow \quad 1 = 4\left(s+\frac{3}{4}\right)A + 4sB$$

Set $s=0$: $\quad A = \frac{1}{3}$

$s = -\frac{3}{4}$: $\quad -3B = 1 \Rightarrow B = -\frac{1}{3}$

So: $\quad X(s) = \frac{1}{3s} - \frac{1}{3}\frac{1}{s+\frac{3}{4}}$

$$\overset{\text{table}}{\Rightarrow} \quad x(t) = \frac{1}{3} - \frac{1}{3}e^{-\frac{3}{4}t}$$

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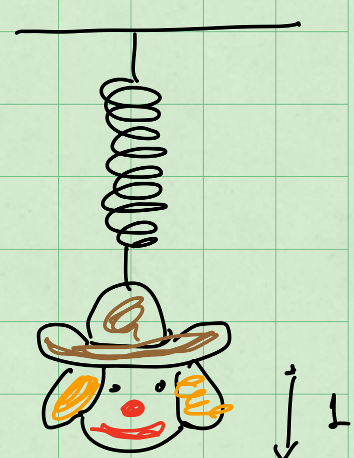
Ex 4:

$$\begin{cases} x'' + 9x = 1 \\ x(0) = 0 \\ x'(0) = 1 \end{cases}$$

Solve w/ Laplace.



$$\frac{s^2+cs+d}{(s-a)(s^2+b^2)} = \frac{A}{s-a} + \frac{Bs+C}{s^2+b^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \quad \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

Sol'n:

$$\mathcal{L}\{x''\} + 9\mathcal{L}\{x\} = \mathcal{L}\{1\}$$

$$\Rightarrow \quad s\mathcal{L}\{x'\} - x'(0)) + 9\mathcal{L}\{x\} = \frac{1}{s}$$

$$\Rightarrow \quad s(s\mathcal{L}\{x\} - x(0)) - x'(0) + 9\mathcal{L}\{x\} = \frac{1}{s}$$

$$\Rightarrow \quad s^2 X(s) - s\underbrace{\cancel{x(0)}}_{=0} \underbrace{-x'(0)}_{=1} + 9X(s) = \frac{1}{s}$$

$$\Rightarrow (s^2+9)X(s) = \frac{1}{s} + 1$$

$$\Rightarrow X(s) = \frac{s+1}{s(s^2+9)} = \underbrace{\frac{A}{s} + \frac{Bs+C}{s^2+9}}$$

$$\frac{s+1}{s(s^2+9)} = \frac{As^2 + 9A + Bs^2 + Cs}{s(s^2+9)}$$

Match terms: $A + B = 0$

$9A = 1$

$C = 1$

$$A = \frac{1}{9}, \quad C = 1, \quad B = -\frac{1}{9}$$

So: $$X(s) = \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9} + \frac{1}{s^2+9}$$

$$= \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+9}$$

$$\Rightarrow x(t) = \frac{1}{9} - \frac{1}{9}\cos(3t) + \frac{1}{3}\sin(3t)$$