Lesson 18 02/21/2022

7.2 Finish, Partial fractions.

Ex 1:
$$\begin{cases} x' + 3x + y = \cos(t) \\ y' - x + 2y = t \end{cases} \quad (*)$$
$x(0) = 0 = y(0)$

Take $\mathcal{L}$ on both sides for both eq's

$$sX(s) - \cancel{x(0)}^{\,0} + 3X(s) + Y(s) = \frac{s}{s^2+1}$$
$$sY(s) - \cancel{y(0)}_{\,0} - X(s) + 2Y(s) = \frac{1}{s^2}$$

$$(s+3)X(s) + Y(s) = \frac{s}{s^2+1}$$
$$-X(s) + (s+2)Y(s) = \frac{1}{s^2} \qquad \times(s+3) \quad (+)$$

$$Y(s) + (s+3)(s+2)Y(s) = \frac{s}{s^2+1} + (s+3)\frac{1}{s^2}$$

$$\Rightarrow \quad Y(s) = \underbrace{\frac{1}{1+(s+3)(s+2)}\left(\frac{s}{s^2+1} + \frac{s+3}{s^2}\right)}_{\text{rational function, can use partial fractions}}$$

Compute $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Once $y(t)$ is found, solve for $x(t)$ in $(*)$. //

Seen: Differentiation $\xrightarrow{\mathcal{L}}$ Multiplication by s

Today Integration $\xrightarrow{\mathcal{L}}$ Multiplication by $\frac{1}{s}$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \quad (*)$$

Taking $\mathcal{L}^{-1}$ on $(*)$:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau = \int_0^t \mathcal{L}^{-1}\{F\}(\tau)\,d\tau$$

Ex: $X(s) = \frac{1}{s(s^2+9)}$, want $\mathcal{L}^{-1}\{X(s)\}$

1st way: partial fractions

2nd:

Set $F(s) = \frac{1}{s^2+9} \Rightarrow X(s) = \frac{F(s)}{s}$

So:

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t \mathcal{L}^{-1}\{F\}(\tau)\,d\tau$$

$$= \int_0^t \frac{1}{3}\sin(3\tau)\,d\tau = -\frac{1}{9}\cos(3\tau)\Big|_0^t$$

$$= \frac{1}{9} - \frac{1}{9}\cos(3t) \quad //$$

7.3 Partial Fractions

Method for decomposing rational functions.

Rational fct: $Q(s) = \dfrac{P(s)}{Q(s)}$,

$P(s)$, $Q(s)$ polynomials, $\deg P < \deg Q$.

Ex: $\dfrac{s^2+3}{s^3+2s+1}$, $\dfrac{s+1}{s^4+2}$

If given $\dfrac{P(s)}{Q(s)}$, $\deg P \geq \deg Q$, use long division.

Ex:

$$F(s) = \frac{s^3 - 3s^2 + 4s - 2}{s^2+1}$$

deg 3

deg 2

$$
\begin{array}{r|l}
 & \quad s \; - \; 3 \\
\hline
s^2+1 & s^3 - 3s^2 + 4s - 2 \\
\ominus & s^3 \qquad\quad + s \\
\hline
 & \quad -3s^2 + 3s - 2 \\
\ominus & \quad -3s^2 \qquad - 3 \\
\hline
 & \qquad\quad 3s + 1
\end{array}
$$

$\deg(3s+1) < \deg(s^2+1)$

$$\Rightarrow \quad s^3 - 3s^2 + 4s - 2 = (s^2+1)(s-3) + \underbrace{3s+1}$$

So: $F(s) = s-3 + \dfrac{3s+1}{s^2+1}$

Now: $\dfrac{P(s)}{Q(s)}$, $\deg P < \deg Q$

1. Factor $Q(s)$ as a product of
   → linear factors: $(s-a)^n$ &
   → irreducible quadratic factors $((s-a)^2+b^2)^m$
   $a, b \in \mathbb{R}$ $b \neq 0$

Notice: linear becomes 0 for $s=a$
irr. does not become 0 for any real s.

Ex:

$(s-1)^2$ → linear factor
(s^2+1) ⟶ irr. quadr. $((s-0)^2+1^2)^1$
$(s^2-1) = (s+1)(s-1)$
linear

2. Part of P.F. decomposition corr. to $(s-a)^n$ is

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

all lower powers appear.

3. Part of P.F.D corr. to $((s-a)^2+b^2)^m$ is

$$\frac{A_1 s + B_1}{((s-a)^2+b^2)} + \frac{A_2 s + B_2}{((s-a)^2+b^2)^2} + \dots + \frac{A_m s + B_m}{((s-a)^2+b^2)^m}$$

Ex 1: $$\frac{\overbrace{s-1}^{\text{deg } 1}}{\underbrace{(s+1)(s^2-s-2)}_{\text{deg } 3}} = F(s)$$

1. factor denominator

$$s^2 - s - 2 = 0 \Rightarrow \quad s=-1,\ s=2 \Rightarrow$$
$$(s^2-s-2) = (s+1)(s-2)$$

So:

$$F(s) = \frac{s-1}{(s+1)^2(s-2)} = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{B_1}{s-2}$$

To find A_1, A_2, B_1: Multiply by denominator

$$s-1 = A_1(s+1)(s-2) + A_2(s-2) + B_1(s+1)^2 \quad (*)$$

To find B_1: set $s=2$:

$$1 = B_1 \cdot 9 \Rightarrow B_1 = \frac{1}{9}$$

A_2: set $s=-1$

$$-2 = A_2(-3) \Rightarrow A_2 = \frac{2}{3}$$

Once A_2, B_1 are known, plug in any s that doesn't make $(s+1)(s-2)=0$.

E.g. $s=1$

$$0 = -2A_1 - \frac{2}{3} + 4\cdot\frac{1}{9}$$

$$\Rightarrow \quad A_1 = -\frac{1}{9}.$$

//

General method: from (*), expand polynomials, match coef. of $1, s, s^2$ on the two sides.

Ex 2: $\dfrac{s-1}{(s+1)(s^2-s+2)}$

Notice: $s^2-s+2 = s^2 - 2\left(\frac{1}{2}\right)s + \frac{1}{4} + \frac{7}{4}$

$$= \underbrace{\left(s-\frac{1}{2}\right)^2 + \frac{7}{4}}_{\text{irr. quadr.}}$$

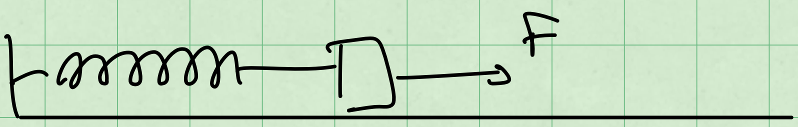
So:

$$\frac{s-1}{(s+1)(s^2-s+2)} = \frac{A_1}{s+1} + \frac{A_2 s + B_2}{s^2-s+2}$$

(compare w/ previous example) //

Ex 3:

Spring-mass system w/ periodic external force



$$\begin{cases} x'' + 9x = 5\cos(\omega t) \\ x(0) = x'(0) = 0 \end{cases}$$

Different behavior depending on whether $\omega \neq \sqrt{9} = 3$ or $\omega = 3$.

Take $\mathcal{L}$:

$$s^2 X(s) - s\,\cancelto{0}{x(0)} - \cancelto{0}{x'(0)} + 9X(s) = 5\,\frac{s}{s^2+\omega^2}$$

$$\Rightarrow X(s) = \frac{5s}{(s^2+\omega^2)(s^2+9)}$$

Inverse Laplace next time.