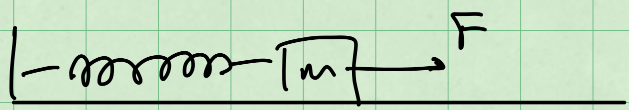
Lesson 19 02/23/22

Finish 7.3

Last time: partial fractions



$$\begin{cases} x'' + 9x = 5\cos(\omega t) \\ x(0) = x'(0) = 0 \end{cases} \qquad \omega \neq 0$$

Examine what happens when $\omega \neq \sqrt{9} = \sqrt{\frac{k}{m}}$ ← spring const (k), ← mass (m)

and when $\omega = \sqrt{9}$.

Computed: $X(s) = \dfrac{5s}{(s^2+\omega^2)(s^2+9)}$

1. $\omega^2 \neq 9$

$$\frac{5s}{(s^2+\omega^2)(s^2+9)} = \frac{A_1 s + B_1}{s^2+\omega^2} + \frac{A_2 s + B_2}{s^2+9}$$

irreducible quadratic

$$5s = (A_1 s + B_1)(s^2+9) + (A_2 s + B_2)(s^2+\omega^2)$$

1st way: Match coefficients

$$5s = A_1 s^3 + 9A_1 s + B_1 s^2 + 9B_1 + A_2 s^3 + A_2\omega^2 s + B_2 s^2 + B_2\omega^2$$

$$\Rightarrow \begin{cases} A_1 + A_2 = 0 \\ 9A_1 + A_2\omega^2 = 5 \\ B_1 + B_2 = 0 \\ 9B_1 + \omega^2 B_2 = 0 \end{cases}$$

4×4 system w/ 4 unknowns

Decouples:

$$\begin{bmatrix} 1 & 1 \\ 9 & \omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 9 & \omega^2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\omega^2 \neq 9$ matrices non-singular, can find A_1, A_2, B_1, B_2. If $\omega^2 = 9$ they are singular.

2nd:

$$5s = (A_1 s + B_1)(s^2 + 9) + (A_2 s + B_2)(s^2 + \omega^2)$$

Set $s = 3i$

$$15i = 0 + (\underbrace{3iA_2}_{\text{imaginary}} + \underbrace{B_2}_{\text{real}})\,(\underbrace{\omega^2 - 9}_{\text{real}})$$

Real & Imaginary pts of both sides must agree.

Re: $\quad 0 = B_2(\omega^2 - 9) \Rightarrow B_2 = 0 \quad (\omega^2 \neq 9)$

Im: $\quad 15 = 3A_2(\omega^2 - 9) \Rightarrow A_2 = \dfrac{5}{\omega^2 - 9} \quad (\omega^2 \neq 9)$

For A_1, B_1: set $s = \omega i$

$$5\omega i = (A_1 \omega i + B_1)(9 - \omega^2)$$

Re: $0 = B_1 (9 - \omega^2) \Rightarrow B_1 = 0$

Im: $5\omega = A_1 \omega (9 - \omega^2) \Rightarrow A_1 = \dfrac{5}{9 - \omega^2}$

$$X(s) = \frac{5}{9 - \omega^2} \frac{s}{s^2 + \omega^2} + \frac{5}{\omega^2 - 9} \frac{s}{s^2 + 9}$$

$$\Rightarrow x(t) = \frac{5}{9 - \omega^2} \underbrace{\cos(\omega t)} - \frac{5}{9 - \omega^2} \underbrace{\cos(3t)}$$

superposition of periodic motions

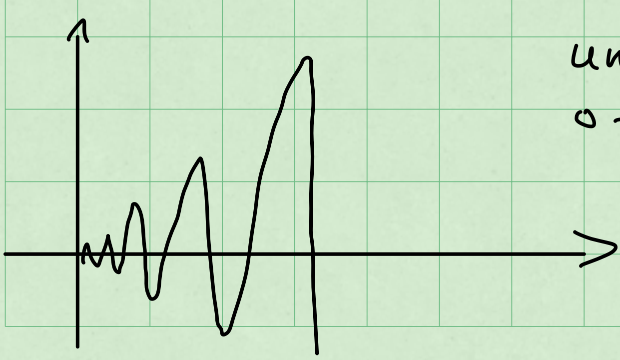
$\omega^2 = 9$:

$$X(s) = \frac{5s}{(s^2 + 9)^2}$$

table

$$\Rightarrow x(t) = \underbrace{\frac{5t}{2 \cdot 3} \sin(3t)}$$

unbounded as a function of t



resonance.

Property: Translation in s axis

If $\mathcal{L}\{f\}$ exists for $s>c$ then $\mathcal{L}\{e^{at}f(t)\}$ exists $s>c+a$ and

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \text{ where } F(s)=\mathcal{L}\{f(t)\}$$

Equivalently:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

Multiplication by exp in $t \xleftrightarrow{\mathcal{L}}$ translation in s.

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = \mathcal{L}^{-1}\{F(s-a)\}$,

$$F(s) = \frac{1}{s}$$

So:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = e^{at}$$

Ex 2: $\mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)^3}\right\}$

1st way: $\frac{s-1}{(s+1)^3} = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3}$

(partial fr)

$$\frac{s-1}{(s+1)^3} = \frac{(s+1)-2}{(s+1)^3} = F(s+1), \quad F(s) = \frac{s-2}{s^3}$$

$$= \frac{1}{s^2} - \frac{2}{s^3}$$

So:

$$\mathcal{L}^{-1}\{F(s+1)\} = e^{-t}\mathcal{L}^{-1}\{F(s)\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{2}{s^3}\right\} =$$

$$= e^{-t}\left(t - t^2\right) \quad //$$

7.4 Convolution.

Look for: a way of combining two functions of t to obtain a third function of t in such a way that:

multiplication

$$\mathcal{L}\{f \boxed{?} g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

Does multiplication work?

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{1\}\mathcal{L}\{1\} = \frac{1}{s^2}$$

If it did, $\frac{1}{s} = \mathcal{L}\{1\} = \mathcal{L}\{1 \cdot 1\} \stackrel{?}{=} \mathcal{L}\{1\}\mathcal{L}\{1\} = \frac{1}{s^2}$

contradiction.

In general: $\mathcal{L}\{fg\} \neq \mathcal{L}\{f\}\mathcal{L}\{g\}$

Define a new operation called <u>convolution</u>

Input: 2 nice enough functions of t

Output: A function of t.

All defined for $t \geq 0$.

$$f * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

convolution

<u>Convolution</u> <u>Theorem</u>: If f, g nice then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}.$$

<u>So:</u> $\mathcal{L}$ turns convolution into multiplication

$$\mathcal{L}^{-1}\{F \cdot G\} = f * g$$