Lesson 2 01/12/2022

Plan:
- Linear systems
- Existence & uniqueness of sols
- Structure of sols
- linear independence.

Linear systems of ODE

$$n \text{ ODES} \begin{cases} x_1'(t) = p_{11}(t)\, x_1(t) + \dots + p_{1n}(t)\, x_n(t) + f_1(t) \\ x_2'(t) = p_{21}(t)\, x_2(t) + \dots + p_{2n}(t)\, x_n(t) + f_2(t) \\ - - - - - - \\ x_n'(t) = p_{n1}(t)\, x_1(t) + \dots + p_{nn}(t)\, x_n(t) + f_n(t) \end{cases}$$

$p_{ij}, f_j \rightarrow$ known functions

$x_1, \dots, x_n$ unknown functions we are looking for.

Ex:
$$\begin{cases} x_1'(t) = \sin(t)\, x_1(t) + e^t\, x_2(t) + \cos(t) \\ x_2'(t) = t^2\, x_1(t) + 5\, x_2(t) \end{cases}$$

$p_{11} = \sin(t)$ $p_{12} = e^t$ $f_1 = \cos(t)$

$p_{21} = t^2$ $p_{22} = 5$ $f_2 = 0$

Non-ex:
$$\begin{cases} x_1'(t) = e^{x_1(t) + x_2(t)} \\ x_2'(t) = 2\, x_2(t) \end{cases}$$

Rewrite linear system using matrix notation,

$$\underline{\underline{x}}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \underline{\underline{P}}(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}, \quad \underline{\underline{f}}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

Write:

$$\underline{\underline{x}}'(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ & & \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}}_{\text{non-homogeneous term}}$$

or: $\underline{\underline{x}}'(t) = \underline{\underline{P}}(t)\, \underline{\underline{x}}(t) + \underline{\underline{f}}(t)$

Ex: $\underline{\underline{P}}(t) = \begin{bmatrix} \sin(t) & e^{t} \\ t^2 & 5 \end{bmatrix}$ $\quad \underline{\underline{f}}(t) = \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \sin(t) & e^{t} \\ t^2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix}$$

Q: Is there a solution for

$$\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}} + \underline{\underline{f}}$$

for given $\underline{\underline{P}}, \underline{\underline{f}}$?

If value $\underline{\underline{x}}(a)$ is specified for some a, is there a unique sol'n w/ this value?

Thm **Existence & Uniqueness of solutions**

→ Let $\underline{\underline{P}}, \underline{\underline{f}}$ have continuous entries on an interval I.

→ Let $a \in I$ (we will specify initial conditions at $t=a$)

→ Let $\underline{\underline{b}}$ be a column vector $(n \times 1)$ (this will be the condition at $t=a$).

Then: $\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}} + \underline{\underline{f}}(t)$ has exactly one solution satisfying $\underline{\underline{x}}(a) = \underline{\underline{b}}$ and this sol'n is defined on all of I.

↳ (specific to linear systems)

Ex: $\circledast$ $\begin{cases} x_1'(t) = \sin(t)\, x_1(t) + \ln(t+1)\, x_2(t) \\ x_2'(t) = x_1(t) + \cos(t)\, x_2(t) + e^t \end{cases}$

Looking for sol'n w/ $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Is there such a sol'n? How many? What is the largest interval on which a solution exists?

In example: $\underline{\underline{P}}(t) = \begin{bmatrix} \sin(t) & \ln(t+1) \\ 1 & \cos(t) \end{bmatrix}$, $\underline{\underline{f}}(t) = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$

$\underline{\underline{f}}(t)$ cont. on $(-\infty, \infty)$

$\underline{\underline{P}}(t)$ cont. on $(-1, \infty)$

So: by thm. there exists exactly one sol'n to the system $\circledast$ w/ $\underline{\underline{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, defined on all of $I = (-1, \infty)$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underset{0}{\Downarrow}$

Q: What do sol's look like? What is their structure?

Observation: superposition principle.

If $\underline{\underline{x}}_1, \dots, \underline{\underline{x}}_n$ are sol's to the system

$$\underline{\underline{x}}' = \underline{\underline{P}}(t)\, \underline{\underline{x}} \underbrace{\qquad\qquad}_{\text{No non-homog. term !!}}$$

then $c_1 \underline{\underline{x}}_1 + \dots + c_n \underline{\underline{x}}_n$ is also a sol'n, if $c_1, \dots, c_n$ are constant scalars.

So: If we know some sol's then we can produce more using linear combinations

$$\underline{\text{Ex:}}\ \circledast \qquad \underline{x}' = \overbrace{\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}}^{\underline{A}} \underline{x}$$

Given the sol'ns $\underline{x}_1(t) = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{x}_2(t) = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(we'll see later how to find them).

$\underline{\text{Check:}}$ $\quad x_1'(t) = 3e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{3t}\begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$$\underline{A}\, x_1(t) = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}\left(e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = e^{3t}\begin{bmatrix} 4-1 \\ -2-1 \end{bmatrix} = e^{3t}\begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

indeed, $\underline{x}_1' = \underline{A}\,\underline{x}_1 \Rightarrow \underline{x}_1$ is a sol'n.

By superposition: $c_1 e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a sol'n for any $c_1, c_2 \in \mathbb{R}$.

Q: How can we know that a given family of sol's can produce <u>any</u> other solution via linear combinations?
How determine what sol's are good building blocks for the entire space of sol's of a system?

$\underline{\text{Ex:}}$ in Ex $\circledast$: $\underline{x}_1(t) = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\tilde{x}}_1(t) = e^{3t}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$
are sol's of $\underline{x}' = \underline{A}\,\underline{x}$ by superposition.

$\underline{\text{But}}$ $\quad \underline{x}_2 = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is also a solution

but it can't be written as $c_1\underline{x}_1(t) + c_2\underline{\tilde{x}}_1(t)$. So:

$\underline{x}_1(t), \tilde{\underline{x}}_1(t)$ are <u>not</u> good building blocks.

<u>Linear Independence</u>

<u>Def'n</u>: The vector valued functions $\underline{f}_1(t), \dots, \underline{f}_n(t)$ defined on an interval I are called <u>linearly independent</u> on I if the following holds:

$$c_1 \underline{f}_1(t) + \dots + c_n \underline{f}_n(t) = 0 \quad \text{for all } t \in I$$
$$\Rightarrow \quad c_1 = \dots = c_n = 0$$

"No non-trivial linear combination can be 0 everywhere on I". Otherwise: <u>linearly dependent</u>.

<u>Ex:</u> $\underline{f}_1(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad \underline{f}_2(t) = \begin{bmatrix} 1 \\ t^2 \end{bmatrix}$

Want to show that $\underline{f}_1, \underline{f}_2$ lin. independent on $\mathbb{R}$.

Let $c_1 \underline{f}_1(t) + c_2 \underline{f}_2(t) = 0$ for all $t \in \mathbb{R}$

$$c_1 \begin{bmatrix} 1 \\ t \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ t^2 \end{bmatrix} = 0 \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 t + c_2 t^2 = 0 \end{cases}$$

for all t.

take e.g. $t = 2$: $\begin{cases} c_1 + c_2 = 0 \\ 2c_1 + 4c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$

so lin. independent.

[<u>note:</u> $f_1(t) - f_2(t)$ vanishes for $t = 1, 0$ but

$\begin{bmatrix} 1 \\ t \end{bmatrix} - \begin{bmatrix} 1 \\ t^2 \end{bmatrix}$ not for all $t \in \mathbb{R}$ so no contradiction]