Lesson 20

02/25/22

Convolution:

$f(t), g(t)$ defined for $t \geq 0$

$$\underset{\overset{\uparrow}{\text{conv}}}{f * g}(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Conv. theorem:

$$\mathcal{L}\{f \underset{\overset{\uparrow}{\text{convolution}}}{*} g\} = \mathcal{L}\{f\} \underset{\overset{\uparrow}{\text{multiplication.}}}{\,} \mathcal{L}\{g\}$$

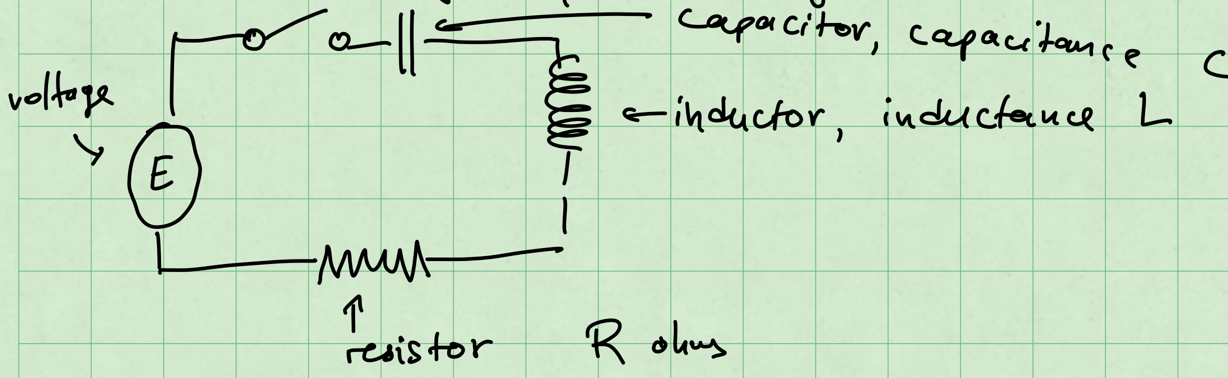
$$\hookrightarrow \quad f * g = \mathcal{L}^{-1}\{F(s)\, G(s)\}$$

Motivation / application

Seen: mass spring systems

$$\underset{\overset{\uparrow}{\text{mass}}}{m}x'' + \underset{\overset{\uparrow}{\text{damping}}}{c}x' + \underset{\overset{\uparrow}{\text{spring const.}}}{k}\overset{\overset{\text{displacement}}{\downarrow}}{x} = \underset{\overset{\uparrow}{\text{external force}}}{f(t)}$$

A mathematically equivalent system: RLC circuit.



Ohm's law:

$$L I' + R \underset{\substack{\uparrow \\ \text{current}}}{I} + \frac{1}{C} \underset{\substack{\uparrow \\ \text{charge in capacitor.}}}{Q} = E(t) \qquad \textcircled{*}$$

but: $I = Q'$ so $\textcircled{*}$

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t)$$

Investigate: output (in terms of charge or current) of RLC circuit corresponding to a given input.

look at $E \rightarrow I$

Assume: $I(0) = 0$, $Q(0) = 0$. Then:

$$Q(t) = \cancelto{0}{Q(0)} + \int_0^t I(\tau)\,d\tau \qquad \left(\text{by } \textcircled{*} \text{ and FTC}\right)$$

$$\textcircled{*} \Rightarrow L I' + R I + \frac{1}{C} \int_0^t I(\tau)\,d\tau = E(t)$$

integrodifferential eq'n

Take $\mathcal{L}$:

$$L\left(s \mathcal{L}\{I\} - \cancelto{0}{I(0)}\right) + R\left(\mathcal{L}\{I\}\right) + \frac{1}{C} \frac{1}{s} \mathcal{L}\{I\} = \mathcal{L}\{E\}$$

$$\mathcal{L}\{I\}\left(Ls + R + \frac{1}{cs}\right) = \mathcal{L}\{E\}$$

$$\mathcal{L}\{I\} = \frac{1}{Ls+R+\frac{1}{cs}}\mathcal{L}\{E\}$$

Set $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{Ls+R+\frac{1}{c}}\right\}$ Impulse Response

So:

$$\mathcal{L}\{I\{ = \mathcal{L}\{h(t)\}\mathcal{L}\{E(t)\}$$

$$\Rightarrow \quad I = h * E$$

<u>Note:</u> $h(t)$ independent of E, depends only on parameters R, L, C of the system. So: if we can determine $h(t)$ then we can find output to any given input via convolution.

e.g. use known voltage E_0, measure I_0.

Then: $\mathcal{L}\{I_0\} = \mathcal{L}\{h(t)\}\mathcal{L}\{E_0(t)\}$

$$\Rightarrow \quad h(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{I_0\}}{\mathcal{L}\{E_0\}}\right\}$$

Find impulse response, can determine

response to any other input.

Property: Convolution is commutative.

$$f * g = g * f$$

Pf: $f * g = \int_0^t f(\tau) g(t-\tau) d\tau =$ $\sigma = t - \tau$

$$= \int_t^0 f(t-\sigma) g(\sigma) (-d\sigma)$$

$$= \int_0^t g(\sigma) f(t-\sigma) d\sigma$$

$$= g * f. \quad //$$

Ex 1: $f(t) = e^t, \; g(t) = t$

$$f * g (t) = \int_0^t e^{\tau} (t-\tau) d\tau$$

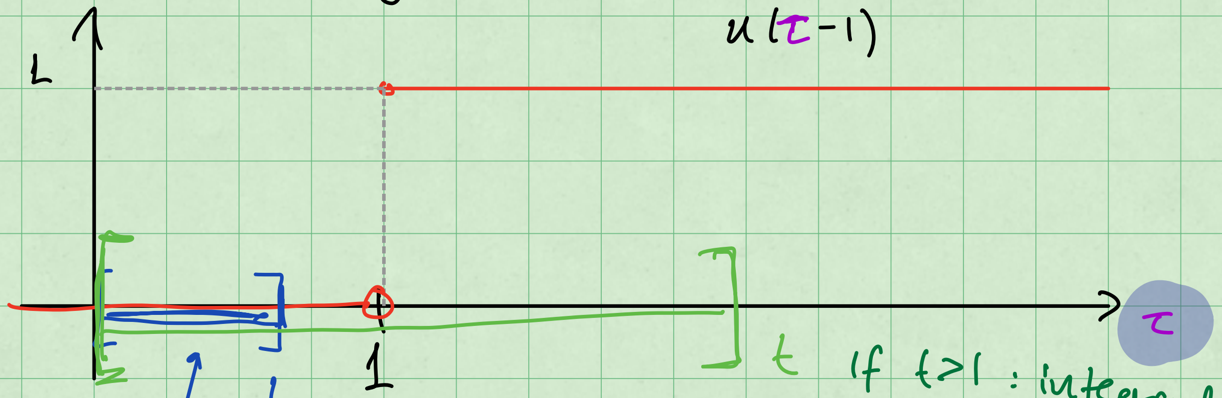
$$= \int_0^t \frac{d}{d\tau}(e^{\tau})(t-\tau) d\tau \overset{IBP}{=} e^{\tau}(t-\tau)\Big|_0^t - \int_0^t e^{\tau}(-1) d\tau$$

$$= -t + e^{\tau}\Big|_0^t = -t + e^t - 1 \quad //$$

Ex 2: $f(t) = u(t-1)$

$g(t) = e^t$

$$f*g\,(t) = \int_0^t u(\tau-1)\, e^{t-\tau}\, d\tau$$



If $t<1$ conv. is integral over this interval.

If $t>1$: integral over green interval

So: if $t<1$: $u(\tau-1) = 0$ on $[0,t]$

$\to f*g\,(t) = 0$

if $t \geq 1$: $u(\tau-1) \begin{cases} = 0 & \text{on } [0,1] \\ = 1 & \text{on } [1,t] \end{cases}$

$$f*g\,(t) = \int_1^t 1 \cdot e^{t-\tau}\, d\tau =$$

← not 0 !

$$= -e^{t-\tau}\Big|_1^t = e^{t-1} - 1$$

So:

$$f * g(t) = \begin{cases} 0, & t < 1 \\ e^{t-1} - 1, & t \geq 1 \end{cases}$$

Notice $f * g$ cont., even though f is not //

Ex 3: $\mathcal{L}^{-1}\left\{ \frac{s}{(s-3)(s^2+1)} \right\}$

1. Partial Fractions.

2. $$\mathcal{L}^{-1}\left\{ \frac{s}{(s-3)(s^2+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s-3} \; \frac{s}{s^2+1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s-3} \right\} * \mathcal{L}^{-1}\left\{ \frac{s}{s^2+1} \right\}$$

$$= e^{3t} * \cos(t)$$

$$= \int_0^t e^{3\tau} \cos(t-\tau)\, d\tau = A$$

$$A = \int_0^t \frac{d}{d\tau}\left(\frac{e^{3\tau}}{3} \right) \cos(t-\tau)\, d\tau$$

$$= \frac{e^{3\tau}}{3} \cos(t-\tau) \Big|_0^t - \int_0^t \frac{e^{3\tau}}{3} \overbrace{\sin(t-\tau)}^{\frac{d}{d\tau}(\cos(t-\tau))} d\tau$$

$$= \frac{e^{3t}}{3} - \frac{1}{3}\cos(t) - \int_0^t \frac{d}{d\tau}\left(\frac{e^{3\tau}}{9}\right)\sin(t-\tau)\,d\tau$$

$$= \frac{e^{3t}}{3} - \frac{1}{3}\cos(t) - \frac{e^{3\tau}}{9}\sin(t-\tau)\Big|_0^t$$

$$+\frac{1}{9}\underbrace{\int_0^t e^{3\tau}\underbrace{\cos(t-\tau)\,d\tau}_{-\frac{d}{d\tau}(\sin(t-\tau))}}_{\overset{\|}{A}}$$

$$A - \frac{1}{9}A = \frac{1}{3}e^{3t} - \frac{1}{3}\cos(t) + \frac{1}{9}\sin(t)$$

$$\Rightarrow \quad A = \frac{9}{8}\left(\frac{1}{3}e^{3t} - \frac{1}{3}\cos(t) + \frac{1}{9}\sin(t)\right) \quad //.$$