Lesson 21 02/28/22

No class next Monday.

7.4 | Seen: Differentiation in t & in integration in t $\longleftrightarrow$ multiplication / division by s

1: $\mathcal{L}\{f'\} = s\,\mathcal{L}\{f\} - f(0)$

2: $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{t}\,\mathcal{L}\{f(t)\}$

Today: Multiplication / division by $t \overset{\mathcal{L}}{\longleftrightarrow}$ differentiation / integration in s.

3. $\mathcal{L}\{(-t)\,f(t)\} = \frac{d}{ds}F(s)$ $\qquad F(s) = \mathcal{L}\{f(t)\}$

4 If $\lim\limits_{t\to 0^+} \frac{f(t)}{t}$ exists & is finite*, then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \underbrace{F(\sigma)\,d\sigma}_{\text{integrates } F \text{ over "large" arguments}}$$

$$f(t) = t\,\mathcal{L}^{-1}\left(\int_s^\infty F(\sigma)\,d\sigma\right)$$

* For the functions we will be working with, enough to check that $f(0)=0$

Why the requirement: $\frac{1}{t} \to \infty$ as $t \to 0^+$ and we need to know that $\frac{f(t)}{t}$

behaves well at 0 in order to take $\mathcal{L}$.

Non-ex: $f(t) = 1$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \underbrace{\int_0^\infty e^{-st}\frac{1}{t}\,dt}_{\text{does not converge.}}$$

Ex 1: $\mathcal{L}\{t^2\cos(2t)\}$

Option 1: Def'n: $\underbrace{\int_0^\infty e^{-st}t^2\cos(2t)\,dt}_{\text{can be done w/ several IBP}}$

2. $\mathcal{L}\{t^2\cos(2t)\} = \mathcal{L}\{(-t)(-t)\cos(2t)\}$

$$= \frac{d}{ds}\mathcal{L}\{(-t)\cos(2t)\} = \frac{d^2}{ds^2}\mathcal{L}\{\cos(2t)\}$$

$$= \frac{d^2}{ds^2}\frac{s}{s^2+4} = \dots = \frac{2s(s^2-12)}{(s^2+4)^3} \quad //$$

Ex 2: $g(t) = \dfrac{e^t - e^{-t}}{t} = 2\dfrac{\sinh(t)}{t}$

$\mathcal{L}\{g(t)\} = ?$

Note: set $f(t) = 2\sinh(t)$, $f(0) = 0$

$$\boxed{\begin{array}{l}\sinh(t) = \dfrac{e^t - e^{-t}}{2} \\ \cosh(t) = \dfrac{e^t + e^{-t}}{2}\end{array}}$$

By defn: $\mathcal{L}\{g\} = \displaystyle\int_0^\infty \underbrace{e^{-st}\frac{2\sinh(t)}{t}\,dt}_{\text{bad.}}$

W/ rule: $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_s^\infty \mathcal{L}\{2\sinh(t)\}(\sigma)\,d\sigma$

$\overset{\text{table}}{=} \displaystyle\int_s^\infty \frac{2}{\sigma^2-1}\,d\sigma \quad \overset{\text{Partial}}{\underset{\text{Fractions}}{=}}$

$= \displaystyle\int_s^\infty \frac{1}{\sigma-1} - \frac{1}{\sigma+1}\,d\sigma$

$= \left(\ln(\sigma-1) - \ln(\sigma+1)\right)\Big|_s^\infty \quad (*)$

Note: $\ln(\sigma-1)\Big|_s^\infty - \ln(\sigma+1)\Big|_s^\infty =$

$\overbrace{\lim_{\sigma\to\infty}\ln(\sigma-1)}^{\infty} - \ln(s-1) - \overbrace{\lim_{\sigma\to\infty}\ln(\sigma+1)}^{\infty} + \ln(s+1)$

indeterminate.

However:

$(*) = \ln\left(\dfrac{\sigma-1}{\sigma+1}\right)\Big|_s^\infty = \lim_{\sigma\to\infty}\ln\dfrac{\sigma-1}{\sigma+1} \to 0$

$- \ln\dfrac{s-1}{s+1}$

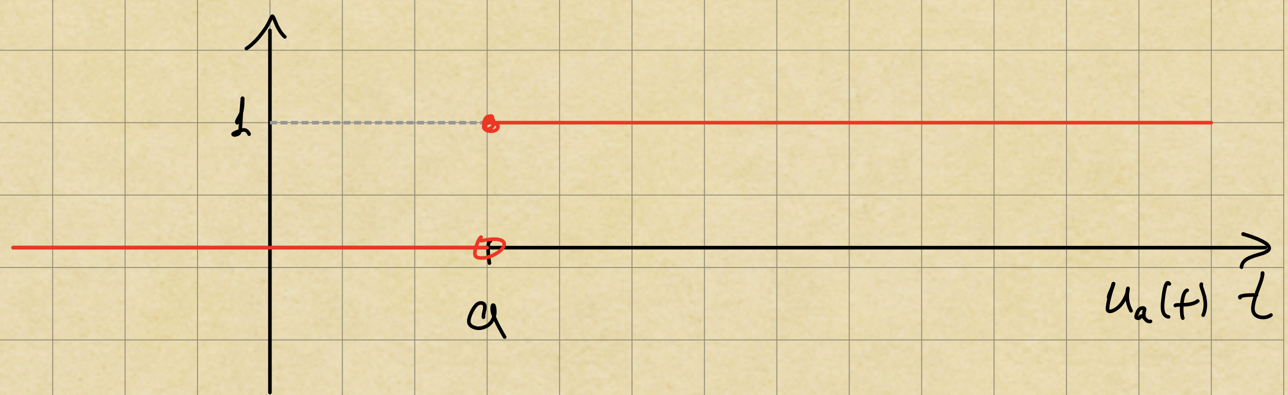
$= -\ln\left(\dfrac{s-1}{s+1}\right)$ //

7.5 Step functions, signals w/ time delay, signals that stop after certain time.

Seen:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$u(t-a) = u_a(t) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$



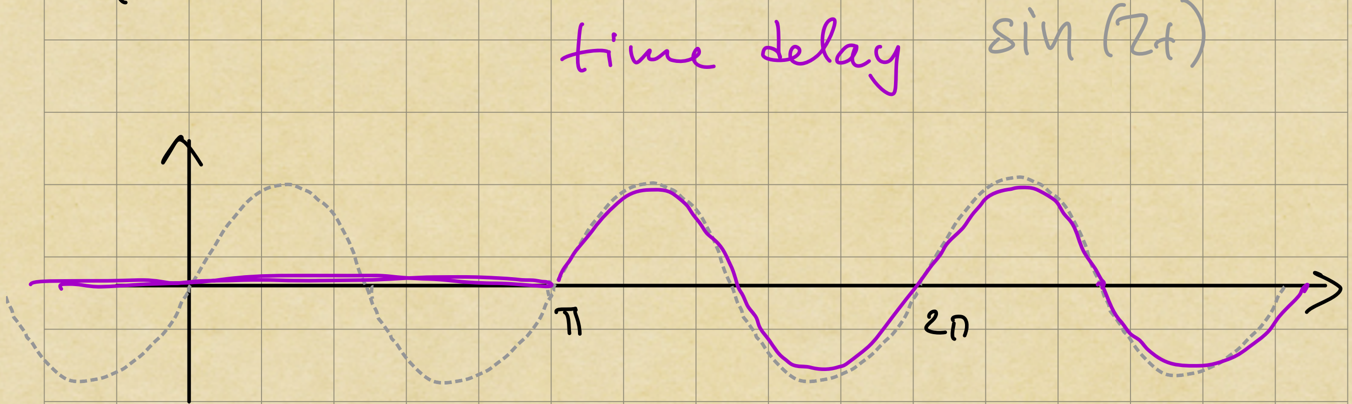
Use step functions to model signals as before

Also consider:

$$1 - u_a(t) = \begin{cases} 0 & t \geq a \\ 1 & t < a \end{cases}$$

Ex 3 $\quad f_1(t) = \begin{cases} \sin(2t) & t \geq \pi \\ 0 & t < \pi \end{cases}$

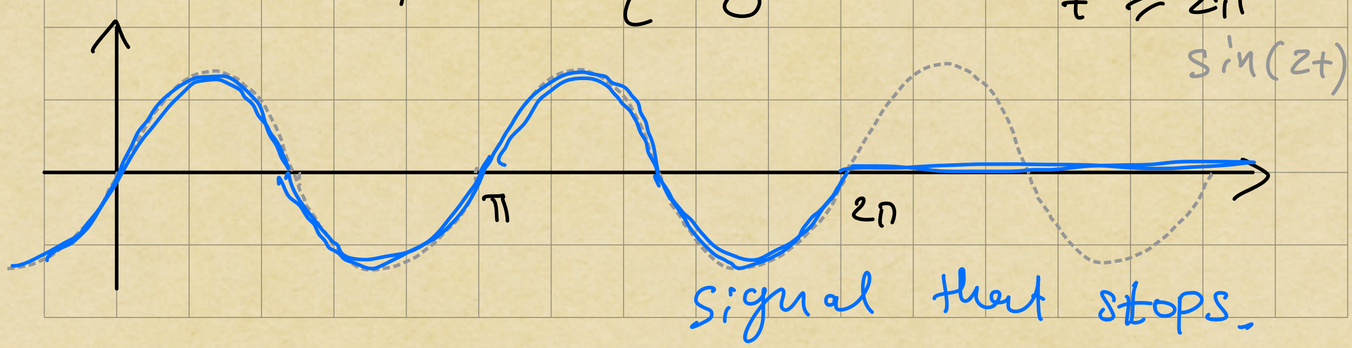
Want: express $f_1(t)$ using step functions.



$$f_1(t) = u(t-\pi)\,\sin(2t)$$

multiply by sth $= 1$ for $t \geq \pi$
$= 0$ for $t < \pi$

Ex 4: $\quad f_2(t) = \begin{cases} \sin(2t) & t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$



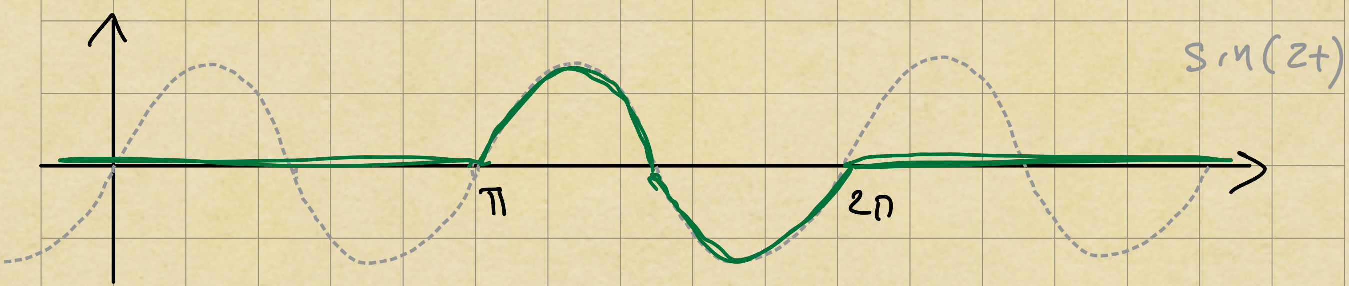
$$f_2(t) = \underbrace{\left(1 - u(t-2\pi)\right)}\sin(2t)$$

multiplies by 1 for $t < 2\pi$

0 for $t \geq 2\pi$

Ex 5:

$$f_3(t) = \begin{cases} \sin(2t), & \pi \leq t < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$



Note:

$$f_3(t) = u(t-\pi)\, f_2(t)$$

$$= \left(1 - u(t-2\pi)\right) f_1(t)$$

$$= \left(1 - u(t-2\pi)\right) u(t-\pi) \sin(2t)$$

$$= \left(u(t-\pi) - u(t-2\pi)\,u(t-\pi)\right) \sin(2t)$$

$$\Rightarrow \boxed{f_3(t) = \left(u(t-\pi) - u(t-2\pi)\right) \sin(2t)}.$$

Check: $u(t-2\pi) = u(t-\pi)\,u(t-2\pi)$

Why it is useful to use step fcts:

Rule: $\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}\,\mathcal{L}\{f(t)\}$

$\Rightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a)$