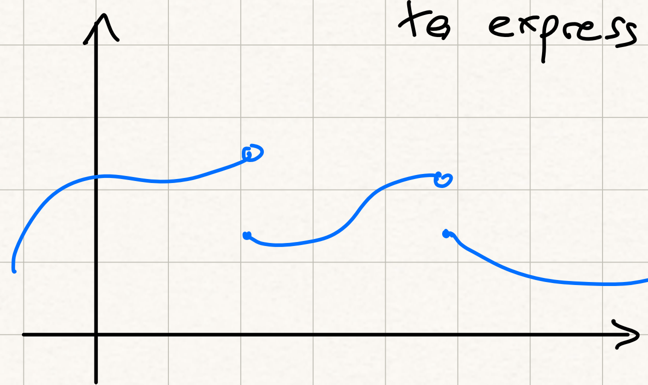
Lesson 22 03/02/22

Last time: Piecewise Cont. functions (ex: signals w/ time delay or signals that stop). Used step functions to express such functions.

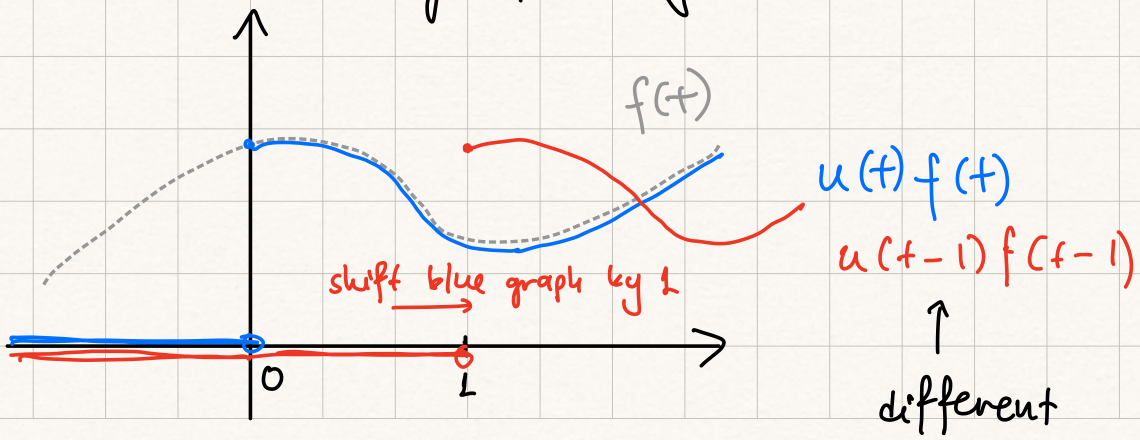


Rule for computing Laplace of piecewise cont. fcts:

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

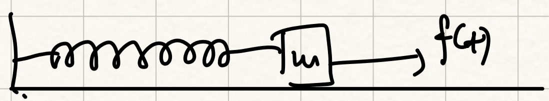
$$\Leftrightarrow \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

Understand graphically $u(t-a)f(t-a)$



$u(t-1)\, f(t)$

Application / Example



Mass-spring, no damping $\quad m = 1, \; k = 9$

$f(t) = \sin(2t)$ external force, starting at $t = \frac{\pi}{3}$, stops at $t = 2\pi$.

$$x'' + 9x = f(t)$$

$$f(t) = \begin{cases} \sin(2t), & \frac{\pi}{3} \le t < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Assume: $\quad x(0) = x'(0) = 0$, want $x(t)$

Steps:
1. Find $\mathcal{L}\{f(t)\}$
   a. Write $f(t)$ using step fcts.
   b. Use rule
2. Find $\mathcal{L}\{x(t)\}$
3. Use same rule to compute $x(t)$

Info: $\quad \mathcal{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \quad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b).$$

1. Find $F(s)$.

$\mathcal{L}\{u(t-a)f(t-a)\}$
$\to e^{-as}F(s)$

a. From last time:

$$f(t) = \left(u(t-\tfrac{\pi}{3}) - u(t-2\pi)\right)\sin(2t)$$

$$= u(t-\tfrac{\pi}{3})\sin(2t) - u(t-2\pi)\sin(2t)$$

$$= u(t-\tfrac{\pi}{3})\sin\left(2(t-\tfrac{\pi}{3}) + \tfrac{2\pi}{3}\right) - u(t-2\pi)\sin\left(2(t-2\pi) + 4\pi\right)$$

$$\Rightarrow \mathcal{L}\{f\} \overset{\text{Rule}}{=} e^{-\frac{\pi}{3}s}\mathcal{L}\left\{\sin\left(2t + \tfrac{2\pi}{3}\right)\right\}$$

$$- e^{-2\pi t}\mathcal{L}\{\sin(2t+4\pi)\}$$

$$= e^{-\frac{\pi}{3}s}\mathcal{L}\left\{\cos\left(\tfrac{2\pi}{3}\right)\sin(2t) + \sin\left(\tfrac{2\pi}{3}\right)\cos(2t)\right\}$$

$$- e^{-2\pi t}\mathcal{L}\{\sin(2t)\}$$

$\uparrow$ sin is 2π-periodic

$$= e^{-\frac{\pi}{3}s}\left(-\frac{1}{2}\right)\frac{2}{s^2+4} + e^{-\frac{\pi}{3}s}\frac{\sqrt{3}}{2}\frac{s}{s^2+4} - e^{-2\pi s}\frac{2}{s^2+4}$$

2. $x'' + 9x = f, \quad x(0) = x'(0) = 0$

$$\xrightarrow{\mathcal{L}} s^2 X(s) + 9X(s) = \mathcal{L}\{f\}$$

$$\Rightarrow X(s) = e^{-\frac{\pi}{3}s} \frac{-1}{(s^2+4)(s^2+9)} + e^{-\frac{\pi}{3}s} \frac{\sqrt{3}\, s}{2(s^2+4)(s^2+9)}$$

$$- e^{-2\pi s} \frac{2}{(s^2+4)(s^2+9)}$$

To find inverse Laplace: Rule

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$$

To find $\mathcal{L}^{-1}\{F(s)\}$: use partial fractions

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{A_1 s + B_1}{s^2+4} + \frac{A_2 s + B_2}{s^2+9}$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{A_1 s + B_1}{s^2+4} + \frac{A_2 s + B_2}{s^2+9}$$