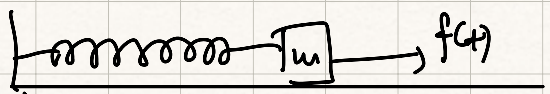
Lesson 23 03/04/2022

No class Monday, No OH M/T

7.5. Last time:

$$\boxed{\begin{aligned} &\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \\ \Leftrightarrow \quad &\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) \end{aligned}}$$



Mass-spring, no damping $m=1$, $k=9$

$f(t) = \sin(2t)$ external force, starting at $t=\frac{\pi}{3}$, stops at $t=2\pi$.

$$x'' + 9x = f(t)$$

$$f(t) = \begin{cases} \sin(2t), & \frac{\pi}{3} \le t < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Assume: $x(0) = x'(0) = 0$, want $x(t)$

Found:

$$X(s) = e^{-\frac{\pi}{3}s}\frac{-1}{(s^2+4)(s^2+9)} + e^{-\frac{\pi}{3}s}\frac{\sqrt{3}\,s}{2(s^2+4)(s^2+9)} - e^{-2\pi s}\frac{2}{(s^2+4)(s^2+9)}$$

To compute $x(t)$: partial fractions

Purple term (similarly for green)

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

$\Rightarrow$

$$\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{3}s}\frac{\sqrt{3}}{2}\frac{s}{(s^2+4)(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{3}s}\frac{\sqrt{3}}{10}\frac{s}{s^2+4}\right\}$$
$$-\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{3}s}\frac{\sqrt{3}}{10}\frac{s}{s^2+9}\right\}$$

Rule

$$= u\left(t-\frac{\pi}{3}\right)\frac{\sqrt{3}}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\left(t-\frac{\pi}{3}\right)$$
$$- u\left(t-\frac{\pi}{3}\right)\frac{\sqrt{3}}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}\left(t-\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{10}u\left(t-\frac{\pi}{3}\right)\cos\left(2\left(t-\frac{\pi}{3}\right)\right)$$
$$- \frac{\sqrt{3}}{10}u\left(t-\frac{\pi}{3}\right)\cos\left(3\left(t-\frac{\pi}{3}\right)\right)$$

Similarly for green terms //

Ex 2: $h(t) = e^t$

$g(t) = u(t-1)$

Want: $h*g$ (did by explicit computation)

$$h*g = \mathcal{L}^{-1}\{\mathcal{L}\{h*g\}\}$$

conv. thm
$$= \mathcal{L}^{-1}\left\{ \mathcal{L}\{h\} \mathcal{L}\{g\} \right\}$$

table
$$= \mathcal{L}^{-1}\left\{ \frac{1}{s-1} \frac{e^{-s}}{s} \right\}$$

Rule
$$= u(t-1)\, \mathcal{L}^{-1}\left\{ \frac{1}{s(s-1)} \right\}(t-1)$$

$$= u(t-1)\, \mathcal{L}^{-1}\left(-\frac{1}{s} + \frac{1}{s-1} \right)(t-1)$$

$$= u(t-1) \left(-1 + e^{t}\right)\Big|_{t-1}$$

$$= u(t-1)\left(-1 + e^{t-1}\right)$$

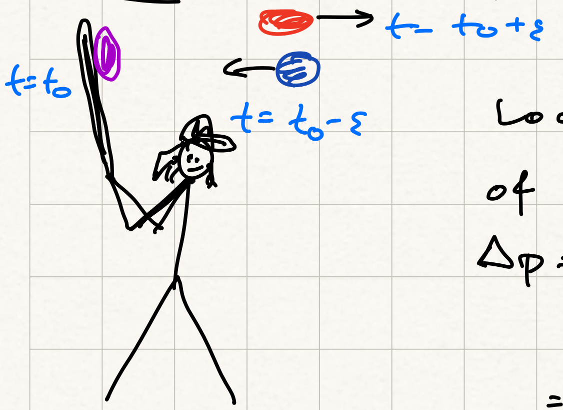
//

If we think of $h(t)$ as the impulse response of a system, $g(t)$ as an input, $h*g$ gives the output cor. to g.

Note: $g = 0$ for $t < 1$ and the output $h*g$ is 0 for $t<1$ as well (principle of causality)

7.6 The delta function

Goal: Model forces acting instantaneously.



$\varepsilon > 0$ small

Look at change in momentum of ball:

$$\Delta p = p\big|_{t_0+\varepsilon} - p\big|_{t_0}$$

$$= mv\big|_{t_0+\varepsilon} - mv\big|_{t_0}$$

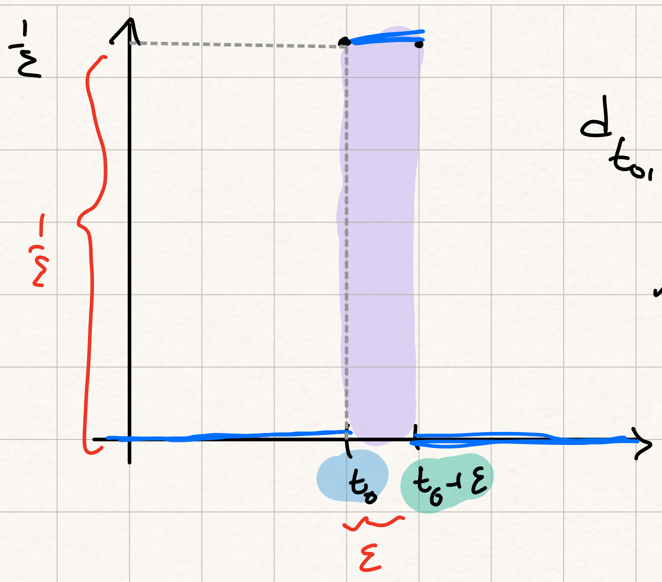
FTC

$$= \int_{t_0}^{t_0+\varepsilon} \frac{d}{dt}(mv)\, dt$$

$$= \int_{t_0}^{t_0+\varepsilon} f(t)\, dt.$$

Observation: to find Δp we don't need the value of f for each time between t_0, $t_0+\varepsilon$, the $\underbrace{\text{integral}}_{\text{impulse of the force.}}$ matters.

So: set up a simple function whose integral over a short period of time is 1 (in order to model a force w/ impulse 1)



$$d_{t_0,\varepsilon} = \begin{cases} \frac{1}{\varepsilon} & t_0 \le t \le t_0+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Note:

$$\int_{t_0}^{t_0+\varepsilon} d_{t_0,\varepsilon}(t)\,dt = \frac{1}{\varepsilon}\cdot\left((t_0+\varepsilon)-t_0\right) = 1$$

To model a force acting instantaneously, want to take $\varepsilon \to 0$. Issue:

$$\lim_{\varepsilon\to 0} d_{t_0,\varepsilon}(t_0) = \infty$$

Limit doesn't make good sense as a function.

We make sense of it as an <u>operator</u>

Dirac delta "function"

Informally: $\delta_{t_0}(t)$ = "$\lim_{\varepsilon\to 0} d_{t_0,\varepsilon}$"

Rigorously: an operator, i.e. a function whose input is a function, output is a number.

input is not real numbers, it is functions.

$$f(t) \longrightarrow \boxed{\delta_{t_0}(t)} \longrightarrow f(t_0)$$

input, a cont. fct

Notation

$$\cancel{\delta_{t_0}(f(t))} = \int_0^\infty f(t)\, \delta_{t_0}(t)\, dt = f(t_0)$$

not an honest integral, bec. δ_{t_0} is not an honest function
just a notation. //

Ex 3:

$$\int_0^\infty 1\, \delta_3(t)\, dt = 1$$

value of const. function 1 at $t=3$

Ex 4:

$$\int_0^\infty \sin(t)\, \delta_{-\frac{\pi}{2}}(t)\, dt = \sin\left(-\frac{\pi}{2}\right) = -1$$

Ex 4: let $a \geq 0$.
$h(t) = e^{-st}$ input, $s > 0$ parameter

$$\int_0^\infty e^{-st}\, \delta_a(t)\, dt = e^{-sa}$$

Recall:

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt$$

Define: $\mathcal{L}\{\delta_a(t)\} := e^{-sa}$

Now we can solve IVP involving impulses.

Ex 5: $\begin{cases} x'' + 4x = \underbrace{5\delta_3(t)} \\ x(0) = x'(0) = 0 \end{cases}$

external force w/ impulse 5 at $t = 3$

Take $\mathcal{L}$ on both sides:

$$s^2 X(s) - \cancel{x'(0)}^{0} - s\,\cancel{x(0)}^{0} + 4X(s) = 5e^{-3s}$$

$$\rightarrow \quad X(s) = \frac{5e^{-3s}}{s^2+4}$$

$$\overset{\text{rule}}{\Rightarrow} \quad x(t) = u(t-3)\, \mathcal{L}^{-1}\left\{\frac{5}{s^2+4}\right\}\Big|_{t-3}$$

$$= u(t-3)\, \frac{5}{2} \sin(2(t-3))$$

Notice: we had impulse at $t = 3$, system remains undisturbed for $t < 3$.

Rmk: if Initial Value Problem involves δ_a, use $\mathcal{L}$.