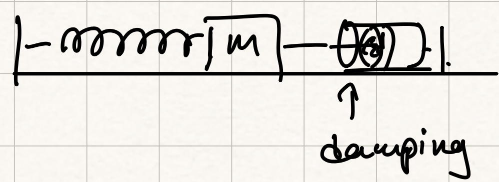
Lesson 24 03/09/22

7.6 **Duhamel's formula**: formula which gives response of a linear system (mass-spring, RLC) to given input

Ex:



$$\begin{cases} m x'' + c x' + k x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

$$m\left(s^2 X(s) - \cancelto{0}{x'(0)} - s\cancelto{0}{x(0)}\right) + c\left(s X(s) - \cancelto{0}{x(0)}\right) + k X(s) = F(s)$$

$$\Rightarrow \quad X(s) = \underbrace{\frac{1}{ms^2 + cs + k}}_{W(s)} F(s) \qquad (*)$$

$W(s)$ transfer function / frequency response

If we set $\underline{w(t) = \mathcal{L}^{-1}\{W(s)\}}$

weight function / unit impulse response

then: $x(t) = w * f$ (by $(*)$)

$$\Rightarrow \quad \boxed{x(t) = \int_0^t w(t-\tau)\, f(\tau)\, d\tau}$$

<u>Duhamel's formula.</u>

<u>Rmk:</u> $w(t)$, $W(s)$ depend only on parameters of system, not the input.

<u>Ex:</u> Spring-mass

$$\begin{cases} x'' + 4x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

Find Duhamel's formula:

$$s^2 X(s) + 4X(s) = F(s)$$

$$\Rightarrow X(s) = \underbrace{\frac{1}{s^2+4}} F(s)$$

$W(s)$: transfer function.

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t).$$

<u>So:</u>

$$x(t) = \int_0^t \frac{1}{2}\sin(2(t-\tau))\, f(\tau)\, d\tau$$

Suppose a force produces impulse 2 at time $t = 4$

$$f(t) = 2\,\delta_4(t)$$

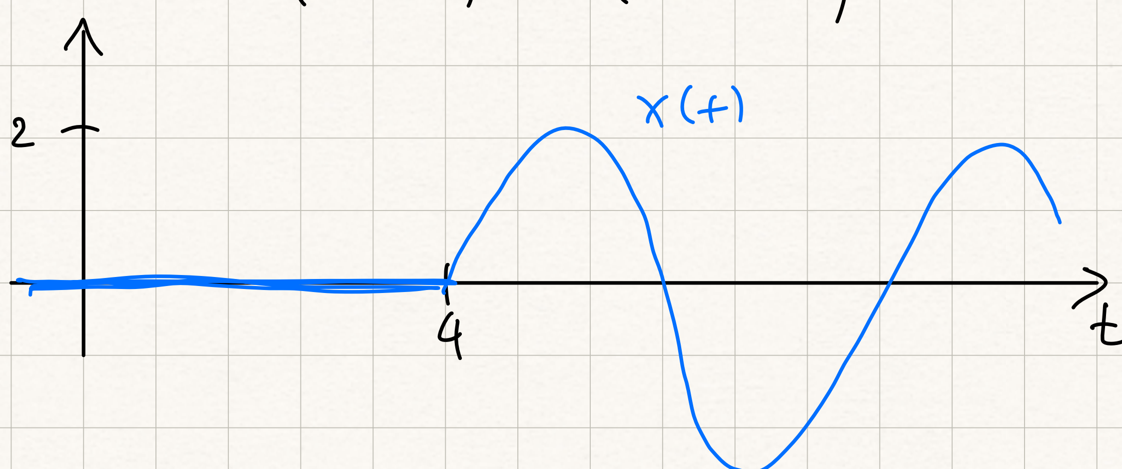
$$x(t) = \int_0^t \frac{1}{2}\sin(2(t-\tau))\; 2\,\delta_4(\tau)\,d\tau$$

$$= \int_0^\infty \big(1 - u(\tau - t)\big)\frac{1}{\cancel{2}}\sin(2(t-\tau))\;\cancel{2}\,\delta_4(\tau)\,d\tau$$

$$h(t) = \begin{cases} 1 & : \tau \le t \\ 0 & : \tau > t \end{cases}$$

$$= \big(\underbrace{1 - u(4-t)}\big)\sin(2(t-4))$$

$$= u(t-4)\sin(2(t-4))$$



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Ch 9: Fourier Series.

9.1 We will see: a method for writing a periodic function as an infinite sum of sines and cosines.

We will use this idea to solve

$$m x'' + kx = \underbrace{f(t)}_{\text{a periodic function}}$$

also: partial diff'l eq'ns

$\partial_t^2 u - \partial_x^2 u$ (wave eq'n in 1 dimensio)

$\partial_t u - \partial_x^2 u$ (heated rod)

Periodic functions

Def'n: A function $f(t)$, $t \in \mathbb{R}$ is called periodic if there exists $p > 0$ such that

$$f(t+p) = f(t)$$

for all $t \in \mathbb{R}$.

Such a $p > 0$ is called a period.

If there exists a smallest period it is called the period.

Ex 1. $\sin(t)$: $\sin(t+2\pi) = \sin(t)$

sin periodic w/ period 2π

Note: If p is a period, $k \cdot p$ is a period for k positive integer.

$2\pi, 4\pi, 6\pi, \ldots$ are periods of sin.

Ex 2: $\left.\begin{array}{l} \cos(nt) \text{ for any } n \\ e^{it} = \cos(t) + i\sin(t) \end{array}\right\}$ periodic

Ex 3: Any constant function is periodic, w/ no smallest period.

Ex 4: $\underbrace{\cos\left(\frac{2\pi}{3}t\right)}_{\mathrm{I}} + \underbrace{\sin(2\pi t)}_{\mathrm{II}}$

I: periodic. To find period:

$$\cos\left(\frac{2\pi}{3}(t+p)\right) = \cos\left(\frac{2\pi}{3}t\right)$$

$$\cos\left(\frac{2\pi}{3}t + \underbrace{\frac{2\pi}{3}p}_{\text{should be a period of } \cos(x)}\right) = \cos\left(\frac{2\pi}{3}t\right)$$

$\frac{2\pi}{3}p = 2\pi \rightarrow p = 3$ is the smallest period.

II: periodic, period 1.

Periods for I: 3, 6, 9, ... $3k$, k int.
II. 1, 2, 3, .. m, m int.

3 is a period for both, so sum is periodic

Ex 4: $\cos\left(\frac{2\pi}{3}t\right) + \sin(t)$
I II

I: periods 3, 6, .. $3k$, k integer.
II: periods 2π, 4π. .. $2m\pi$, m integer.

Is the sum periodic? try to find period which works for both

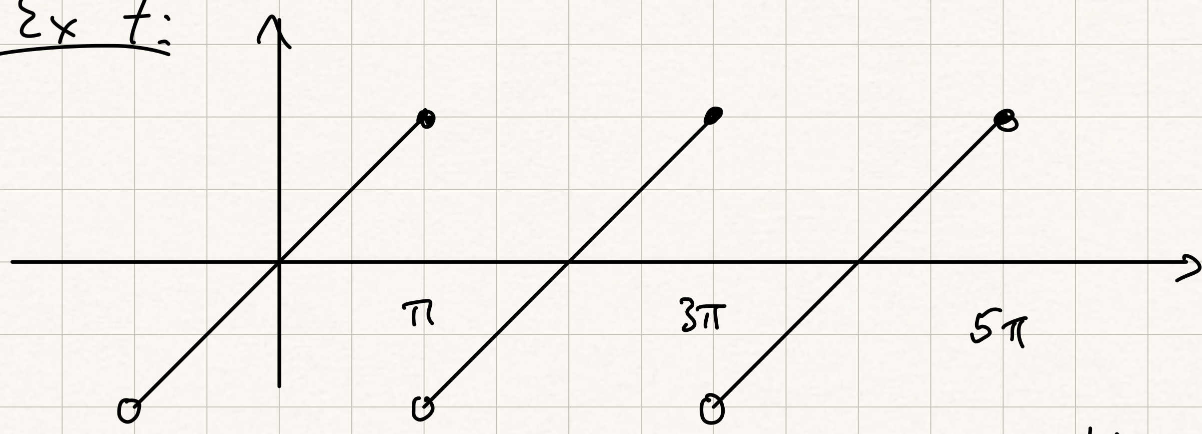
$$3k = 2m\pi \qquad k, m \text{ integers}$$

$\Rightarrow \pi = \frac{3k}{2m}$ can't happen bec. π is irrational.

So sum is <u>not</u> periodic.

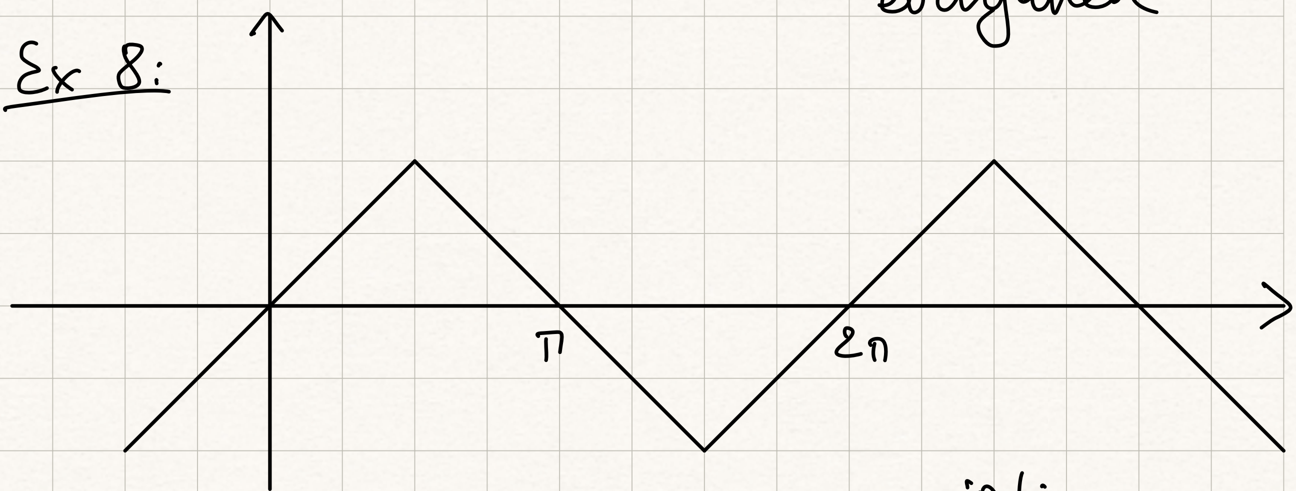
<u>Ex 6:</u> $\arctan(t)$, $\sinh(t)$, $\cosh(t)$ are <u>not</u> periodic.

Ex 7:



periodic,
not continuous
everywhere

Ex 8:



periodic
cont, not
differentiable
everywhere

Note: Ex 1 – 4 were periodic functions written as finite sums of sines, cosines & constants.
7 & 8 can't be written as finite sums of sines, cosines & constants

because they are not continuous/ diff'ble.

Fourier's approach: write periodic functions as <u>infinite</u> sums of trig. fcts.

If f 2π-periodic, we will write it as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\underbrace{nt}) + b_n \sin(\underbrace{nt}) \right)$$

period $\frac{2\pi}{n}$