Lesson 25

03/11/2022

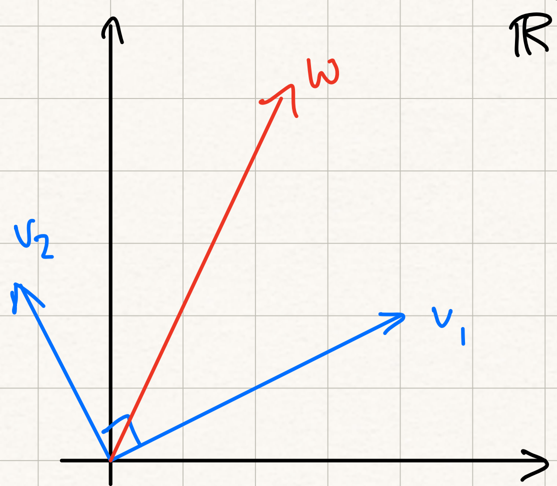
Last time: hoped to write a 2π-periodic function as

$$f \overset{?}{=} a_0 \frac{1}{2} + \sum_{n=1}^{\infty} \left(a_n \underbrace{\cos(nt)} + b_n \underbrace{\sin(nt)} \right)$$

$\frac{2\pi}{n}$ periodic $\Rightarrow$ 2π-periodic

Today: determine a_0, a_n, b_n in terms of the given f.

Motivation:



$\mathbb{R}^2$

v_1, v_2 known vectors, $v_1 \cdot v_2 = 0$

w given, want to write

$$w = a_1 v_1 + a_2 v_2 \quad (*)$$

for some a_1, a_2 unknown, tbd

$(*) \xrightarrow{\cdot v_1}$ $w \cdot v_1 = a_1 \, v_1 \cdot v_1 + a_2 \cancel{v_2 \cdot v_1}^{\,0}$

$\Rightarrow \quad a_1 = \dfrac{w \cdot v_1}{v_1 \cdot v_1} = \dfrac{w \cdot v_1}{|v_1|^2}$

Similarly: $a_2 = \dfrac{w \cdot v_2}{|v_2|^2}$ //

Want: $f = a_0 \frac{1}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$

Analogies: $f \leftrightarrow w$

$a_0, a_n, b_n \leftrightarrow a_1, a_2$

$\frac{1}{2}, \cos(nt), \sin(nt) \leftrightarrow v_1, v_2$

Important ingredient: $v_1 \cdot v_2 = 0$.

Analog of dot product: $\cancel{f \text{ "}\cdot\text{" } g} = \int_{-\pi}^{\pi} f(t) g(t)\, dt$

Def'n: 2 functions $u(t), v(t)$ defined on $[a,b]$ are called orthogonal on $[a,b]$ if

$$\int_a^b \underbrace{u(t)\, v(t)}_{\text{usual multiplication.}} dt = 0$$

Ex 1: a) $u(t) = 1$, $v(t) = \cos t$ $\quad [a,b] = [-\pi, \pi]$

$$\int_{-\pi}^{\pi} 1 \cdot \cos t\, dt = \sin t \Big|_{-\pi}^{\pi} = 0$$

$\Rightarrow$ $1, \cos(t)$ orthogonal on $[-\pi, \pi]$.

❗ Interval is important.

$$\int_{-\pi}^{\frac{3\pi}{2}} 1 \cdot \cos t \, dt = \sin t \Big|_{-\pi}^{\frac{3\pi}{2}} = -1$$

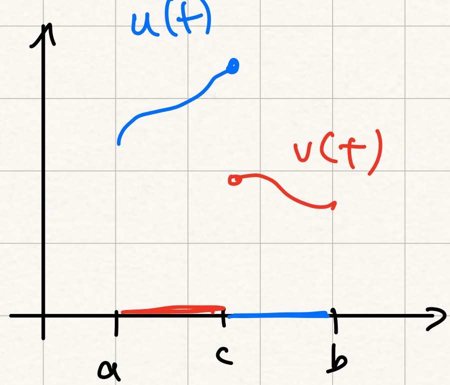
$1, \cos t$ <u>not</u> orthogonal on $\left[-\pi, \frac{3\pi}{2}\right]$.

b) $u(t) = \cos(t)$, $v(t) = \sin(t)$

$[a,b] = [-\pi, \pi]$

$$\int_{-\pi}^{\pi} \cos(t)\sin(t)\, dt = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2t)\, dt = \ldots = 0$$

c)



$$\int_{a}^{b} u(t)\, v(t)\, dt = 0$$

<u>Fact:</u> (proof: exercise)

$n, m = 1, 2, \ldots$

a) $$\int_{-\pi}^{\pi} \cos(mt) \cos(nt)\, dt = \begin{cases} 0 & n \neq m \\ \pi, & n = m \end{cases}$$

"cosines of different frequencies are orthogonal on $[-\pi, \pi]$"

b) $\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$

c) $\int_{-\pi}^{\pi} \sin(mt)\cos(nt)\,dt = 0 \qquad m, n$

d) $\int_{-\pi}^{\pi} \cos(mt)\cdot 1\,dt = \int_{-\pi}^{\pi} \sin(mt)\cdot 1\,dt$

$= 0$

Assumptions: f piecewise continuous
2π - periodic
It has a Fourier series
which can be integrated
term by term.

$$f = \frac{a_0}{2}\cdot 1 + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

To find a_0: Multiply by 1 and integrate.

$$\int_{-\pi}^{\pi} f(t)\cdot 1\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi} 1\cdot 1\,dt + \sum_{n=1}^{\infty} a_n \cancel{\int_{-\pi}^{\pi} \cos(nt)\cdot 1\,dt}^{\,0}$$

$$+ \sum_{n=1}^{\infty} b_n \cancel{\int_{-\pi}^{\pi} \sin(nt)\cdot 1\,dt}^{\,0}$$

by fact

$$\Rightarrow \quad \frac{a_0}{2} \underbrace{\int_{-\pi}^{\pi} 1 \, dt}_{2\pi} = \int_{-\pi}^{\pi} f(t)\, dt$$

$$\Rightarrow \quad \boxed{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt}$$

To find a_m for some fixed m: Multiply by $\cos(mt)$, integrate

$$\int_{-\pi}^{\pi} f(t) \cos(mt)\, dt = \frac{a_0}{2} \int_{-\pi}^{\pi} 1 \cdot \cos(mt)\, dt \to 0 \quad \text{fact}$$

$$+ \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nt) \cos(mt)\, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nt) \cos(mt)\, dt \to 0$$

$$= \sum_{\substack{n=1 \\ m \neq n}}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nt) \cos(mt)\, dt \to 0 \quad (m \neq n)$$

$$\underbrace{+ a_m \int_{-\pi}^{\pi} \cos^2(mt)\, dt}_{\text{isolate } m\text{-term}} \overset{\text{by fact}}{=} \pi$$

$$\Rightarrow \int_{-\pi}^{\pi} f(t)\cos(mt) = a_m \pi$$

$$\Rightarrow \boxed{a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(mt)\,dt}$$

Similarly:

$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)\,dt}$$

Logic of what we did: if f well behaved and is equal to its series expansion then the coefficients have to be as above.

Regardless of whether this is true, we will *define* the Fourier series of a piecewise cont., 2π-periodic function as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

the F.S. of f.

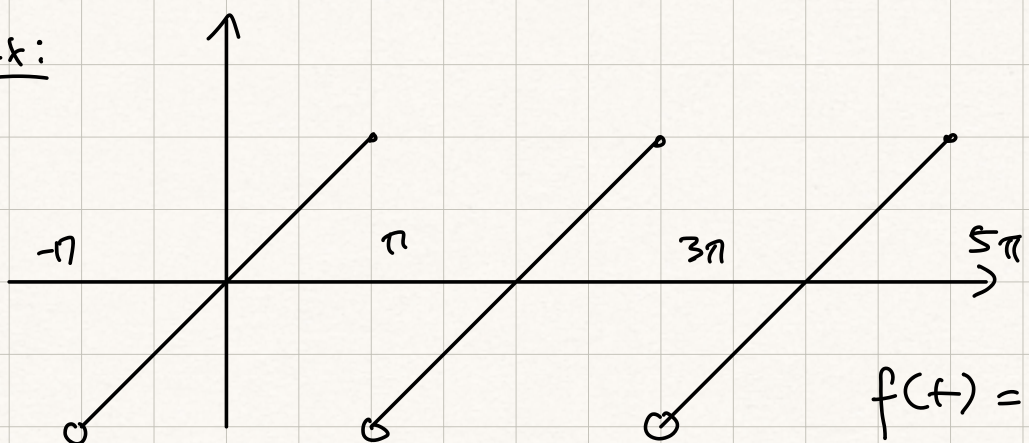
w/ a_0, a_n, b_n as above.

Write:

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

meaning: a_0, a_n, b_n are given by the boxed formulas in terms of f, no claim that f is equal to infinite sum.

Ex:



$f(t) = t$ on $(-\pi, \pi]$

Compute F.S.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t \, dt = \frac{1}{\pi} \left. \frac{t^2}{2} \right|_{-\pi}^{\pi} = 0$$

Exercise: if $n \geq 1$ $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt$$

$$= -\frac{1}{\pi n}\int_{-\pi}^{\pi} t\,\big(\cos(nt)\big)'\,dt$$

$$= -\frac{1}{\pi n}\big(t\cos(nt)\big)\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nt)\,dt$$

$$= -\frac{1}{\pi n}\big(\pi\cos(n\pi) + \pi\cos(n\pi)\big) + \underbrace{\frac{1}{n^2\pi}\sin(nt)\Big|_{-\pi}^{\pi}}_{0}$$

$$= -\frac{2}{n}\cos(n\pi) = -\frac{2}{n}(-1)^n.$$