Lesson 26 03/21/22

Fourier series for 2π-periodic fcts.

If f 2π-periodic

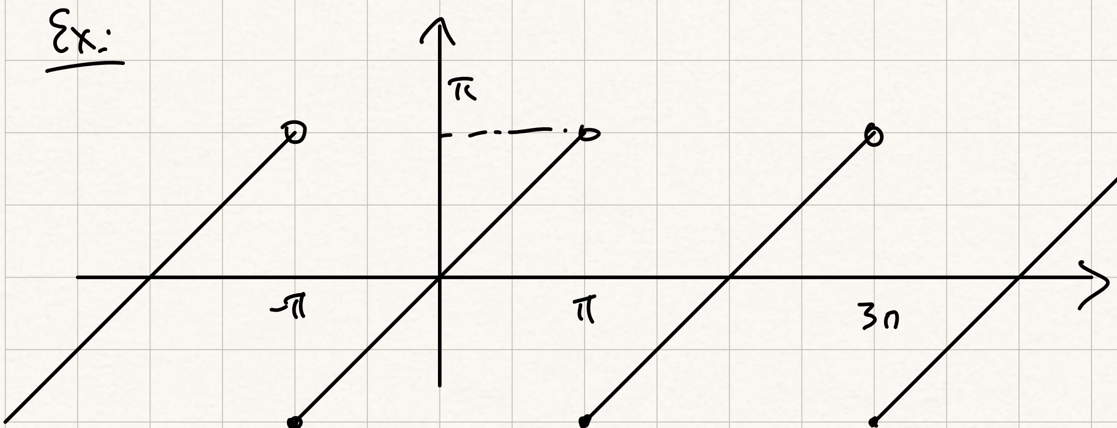
$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos(\underbrace{nt}_{\frac{2\pi}{n}\text{-periodic}}) + b_n \sin(nt)\Big)$$

$\Longrightarrow$ also 2π periodic

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \,, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt\,.$$

Ex:

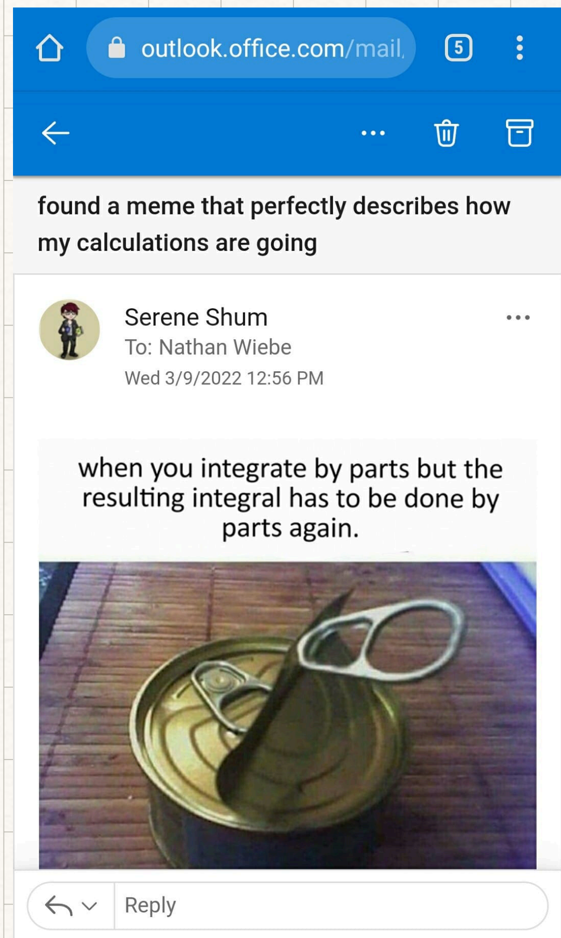


Found: $a_0 = 0, \quad a_n = 0$

$$b_n = -\frac{2}{n}\underbrace{\cos(n\pi)}_{(-1)^n} = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

Fourier series:

$$f \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nt)$$



9.2 More generally: find Fourier Series expansions for functions which are periodic w/ period $P = 2L$

L: half period

Ex:

$\sin(\pi t)$ periodic w/ period $P = 2$

$L = 1$

<u>Want:</u> F.S. for a P-periodic fct $f(t)$.

Know F.S. for 2π-periodic functions

<u>Consider:</u> $g(u) := f\left(\frac{L}{\pi}u\right)$

<u>Claim:</u> g 2π-periodic

$$g(u+2\pi) = f\left(\frac{L}{\pi}(u+2\pi)\right) = f\left(\frac{L}{\pi}u + \overbrace{2L}^{P}\right)$$

$$= f\left(\frac{L}{\pi}u + P\right) \underset{f\ P\text{-periodic}}{=} f\left(\frac{L}{\pi}u\right) = g(u)$$

Know how to write F.S. for g:

$$g(u) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(nu) + b_n \sin(nu)\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\,du \qquad \underbrace{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos(nu)\,du}_{\mathrm{I}}$$

$$\mathrm{II}\ \underbrace{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin(nu)\,du.}$$

Set $t = \frac{L}{\pi}u \Rightarrow f(t) = g\left(\frac{\pi t}{L}\right)$

$$u = \frac{\pi t}{L}$$

<u>So:</u>

$$\boxed{f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)\right)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Lu}{\pi}\right) du$$

$u = \frac{\pi t}{L}$

$$\underset{du = \frac{\pi}{L}dt}{=} \frac{1}{\pi}\int_{-L}^{L} f(t)\,\frac{\pi}{L}\,dt = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

So:

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt}$$

From I & II deduce:

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right) dt}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right) dt}$$

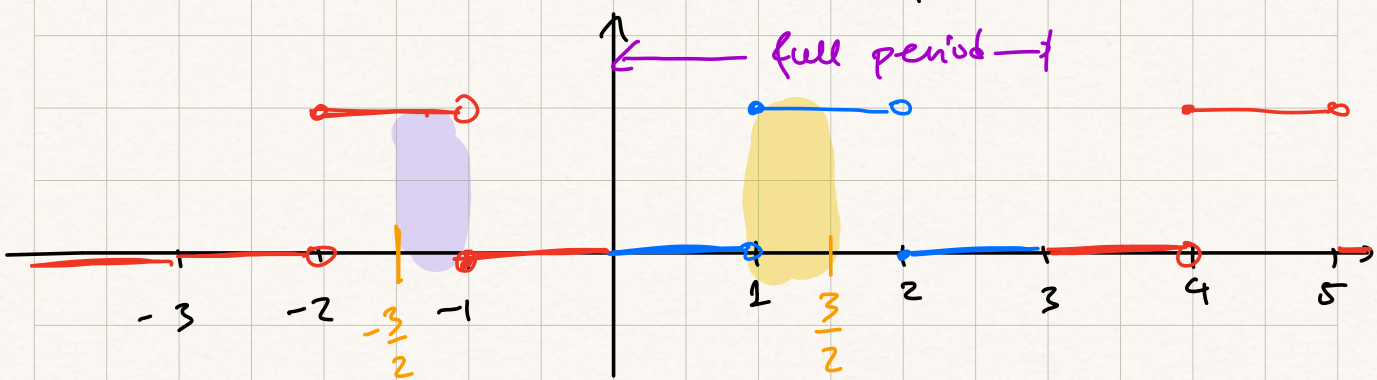
So: boxed expressions : F.S. for $P = 2L$ -

periodic fct.

Note: if $P = 2\pi \Rightarrow L = \pi$ we find the formulas in the beginning of class.

Ex: $f(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & 0 \le t < 1, \quad 2 \le t < 3 \end{cases}$

Periodic w/ period 3 : $f(t+3) = f(t)$



Period: $P = 3 \Rightarrow L = \frac{3}{2}$

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(n \frac{2\pi}{3} t\right) + b_n \sin\left(n \cdot \frac{2\pi}{3} t\right) \right)$$

Find: a_0, a_n, b_n

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t)\,dt = \frac{2}{3} \int_{-\frac{3}{2}}^{\frac{3}{2}} f(t)\,dt$$

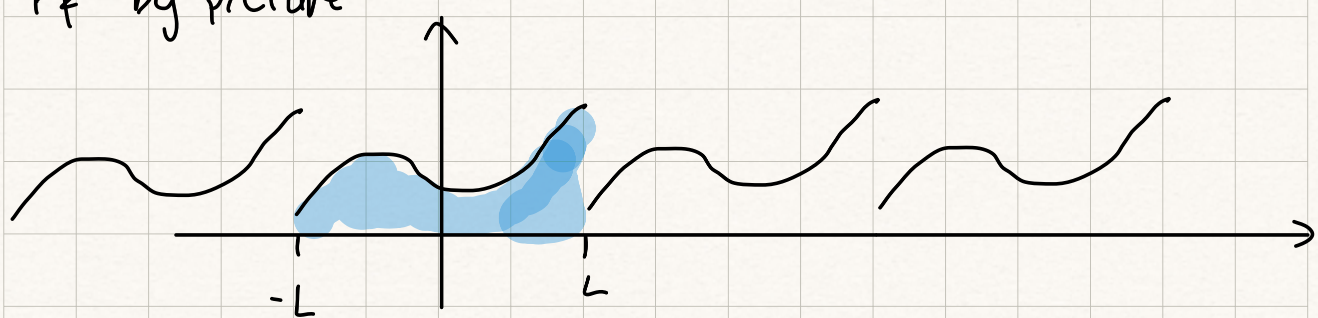
$$= \frac{2}{3}\left(\int_{-\frac{3}{2}}^{-1} 1\,dt + \int_{1}^{\frac{3}{2}} 1\,dt\right) \ldots$$

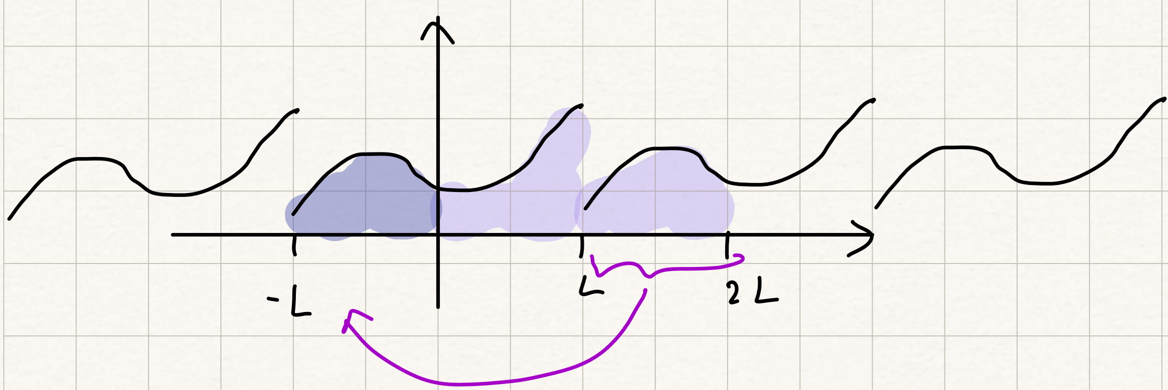
Nicer way:

For a 2L-periodic function

$$\int_{-L}^{L} f\,dt = \int_{0}^{2L} f\,dt = \int_{a}^{a+2L} f\,dt$$

"Pf by picture"





In our case: easier

$$a_0 = \frac{1}{L}\int_{0}^{2L} f(t)\,dt = \frac{2}{3}\int_{0}^{3} f(t)\,dt$$

$$= \frac{2}{3} \int_1^2 1 \, dt = \frac{2}{3}$$

Similarly:

$$a_n = \frac{1}{L} \int_0^{2L} f(t) \cos\left(\frac{n\pi}{L} t\right) dt$$

$$= \frac{2}{3} \int_0^3 f(t) \cos\left(\frac{2n\pi}{3} t\right) dt$$

$$= \frac{2}{3} \int_1^2 1 \cdot \cos\left(\frac{2n\pi}{3} t\right) dt$$

$$= \frac{2}{3} \frac{3}{2\pi n} \sin\left(\frac{2n\pi}{3} t\right) \Big|_1^2$$

$$= \frac{1}{\pi n} \left(\sin\left(\frac{4\pi}{3} n\right) - \sin\left(\frac{2\pi}{3} n\right) \right)$$

$$b_n = \frac{1}{\pi n} \left(\cos\left(\frac{2}{3} \pi n\right) - \cos\left(\frac{4}{3} \pi n\right) \right)$$

Next time: simplify a_n, b_n (actually: $b_n = 0$)