Lesson 27 03/23/2022

9.2

Last time: Fourier Series of 2L-periodic functions.

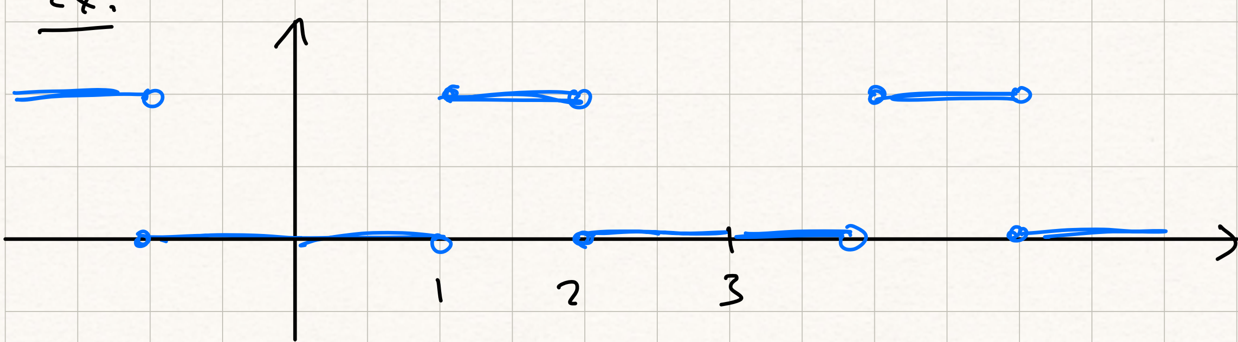
$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{\pi n}{L}t\right) + b_n \sin\left(\frac{\pi n}{L}t\right)\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{1}{L}\int_{0}^{2L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt.$$

Ex:



Found: $a_0 = \frac{2}{3}$, $a_n = \frac{1}{n\pi}\left(\sin\left(\frac{4\pi}{3}n\right) - \sin\left(\frac{2\pi}{3}n\right)\right)$

$$b_n = \frac{1}{\pi n}\left(\cos\left(\frac{2\pi}{3}n\right) - \cos\left(\frac{4\pi}{3}n\right)\right)$$

Simplify:

$$a_n = \frac{1}{n\pi}\Big(\underbrace{\sin\left(\frac{4\pi}{3}n\right) - \sin\left(\frac{2\pi}{3}n\right)}_{=:\tilde{a}_n}\Big) = \frac{1}{n\pi}\tilde{a}_n$$

$\sin\left(\frac{4\pi}{3}x\right) \to$ period $\frac{3}{2}$, so 3 is also a period.

So: $\sin\left(\frac{4\pi}{3}(n+3)\right) = \sin\left(\frac{4\pi}{3}n\right)$

Also: $\sin\left(\frac{2\pi}{3}x\right)$ periodic w/ period 3

So: $\sin\left(\frac{2\pi}{3}(n+3)\right) = \sin\left(\frac{2\pi}{3}n\right)$

Therefore: $\tilde{a}_n = \sin\left(\frac{4\pi}{3}n\right) - \sin\left(\frac{2\pi}{3}n\right) \Rightarrow$

$\tilde{a}_{n+3} = \tilde{a}_n$

So what: $a_4 = \frac{1}{4\pi}\tilde{a}_4 = \frac{1}{4\pi}\tilde{a}_1$

$a_5 = \frac{1}{5\pi}\tilde{a}_5 = \frac{1}{5\pi}\tilde{a}_2$

...

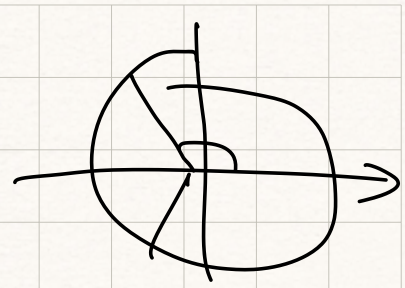
So: enough to compute $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ and $\tilde{a}_n$ is one of them for $n = 4, 5, 6, \ldots$

So: $\tilde{a}_1 = \sin\left(\frac{4\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right) =$

$= 2\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \tilde{a}_1 = -\sqrt{3}$

$$\tilde{a}_2 = \sin\left(\frac{8\pi}{3}\right) - \sin\left(\frac{4\pi}{3}\right)$$

$$= \sqrt{3}$$



$$\tilde{a}_3 = \sin(4\pi) - \sin(2\pi) = 0$$

So:

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{1}{\pi n}\tilde{a}_n = \begin{cases} \frac{1}{\pi n}\overbrace{(-\sqrt{3})}^{\tilde{a}_1} & \text{if } n = 3k+1 \\ \frac{1}{\pi n}\underbrace{\sqrt{3}}_{\tilde{a}_2} & \text{if } n = 3k+2 \\ 0 & \text{if } n = 3k+3 \end{cases}$$

$k = 0, 1, 2, \ldots$

For b_n:

$$b_n = \frac{1}{\pi n}\Big(\underbrace{\cos\left(\frac{2\pi}{3}n\right) - \cos\left(\frac{4\pi}{3}n\right)}_{\tilde{b}_n}\Big)$$

check: $\tilde{b}_{n+3} = \tilde{b}_n$

$$\tilde{b}_1 = \cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{4\pi}{3}\right) = 0$$

$$\tilde{b}_2 = \cos\left(\frac{4\pi}{3}\right) - \cos\left(\frac{8\pi}{3}\right) = 0$$

$$\tilde{b}_3 = \cos(2\pi) - \cos(4\pi) = 0$$

So: $\tilde{b}_n = 0$ for all n, so $b_n = 0$ for all n.

Convergence of Fourier Series

What we've done: took a $2L$-periodic function, computed a series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right) \right) \quad (*)$$

where a_n, b_n are computed from f as above.

Q: Does $(*)$ converge to a number for every t? Is this number equal to $f(t)$?

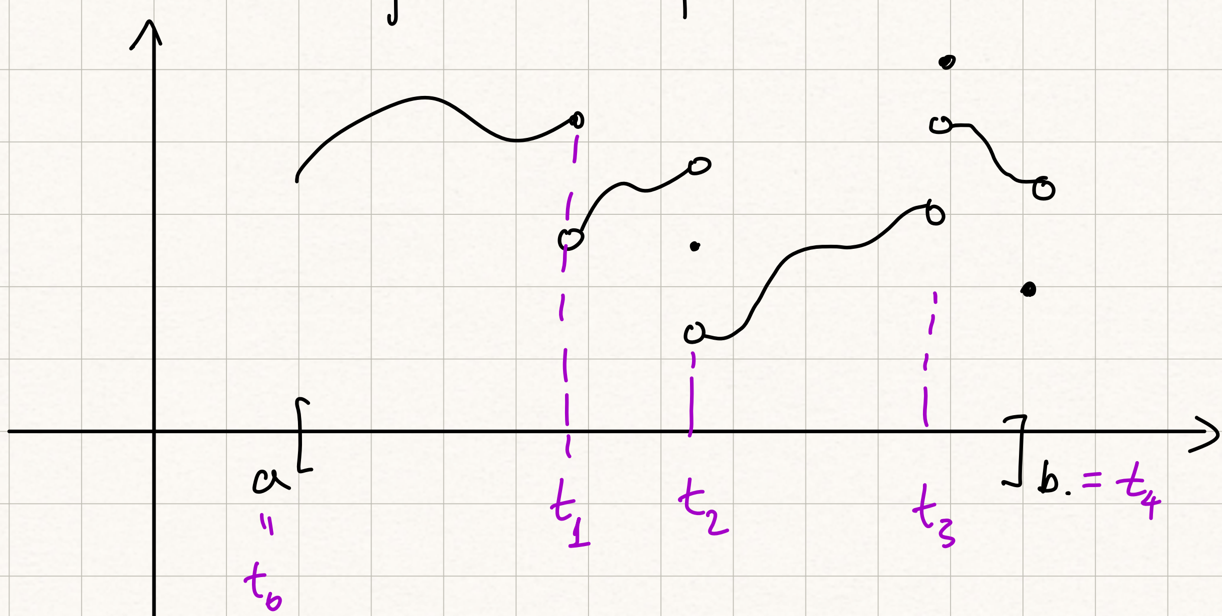
Want: for each t

$$\lim_{N \to \infty} \left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right) \right) = f(t)$$

What happens: true for "most" t under suitable assumptions on f.

Def'n f is piecewise continuous on an interval $[a,b]$ if there are
$$a = t_0, t_1, \dots, t_{n+1} = b$$
where $t_0, t_1, \dots, t_{n+1} \in [a,b]$ such that

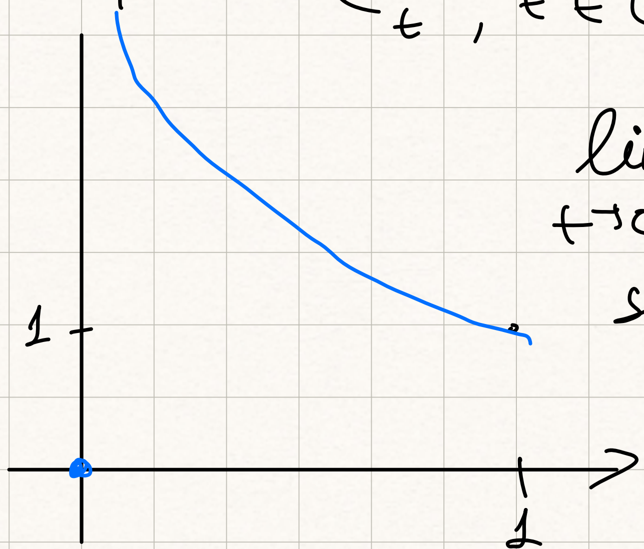
AND
→ f cont. on (t_j, t_{j+1}) $j = 0, \dots, n$
→ $\lim_{t \to t_j^{\pm}} f(t)$ exist and are finite.



On (t_0, t_1), (t_1, t_2), (t_2, t_3), (t_3, t_4) cont.
Side limits exist at t_j and are finite.

Non-example:

$$f(t) = \begin{cases} 0, & t=0 \\ \frac{1}{t}, & t \in (0,1] \end{cases}$$



$$\lim_{t \to 0^+} f(t) = \infty$$

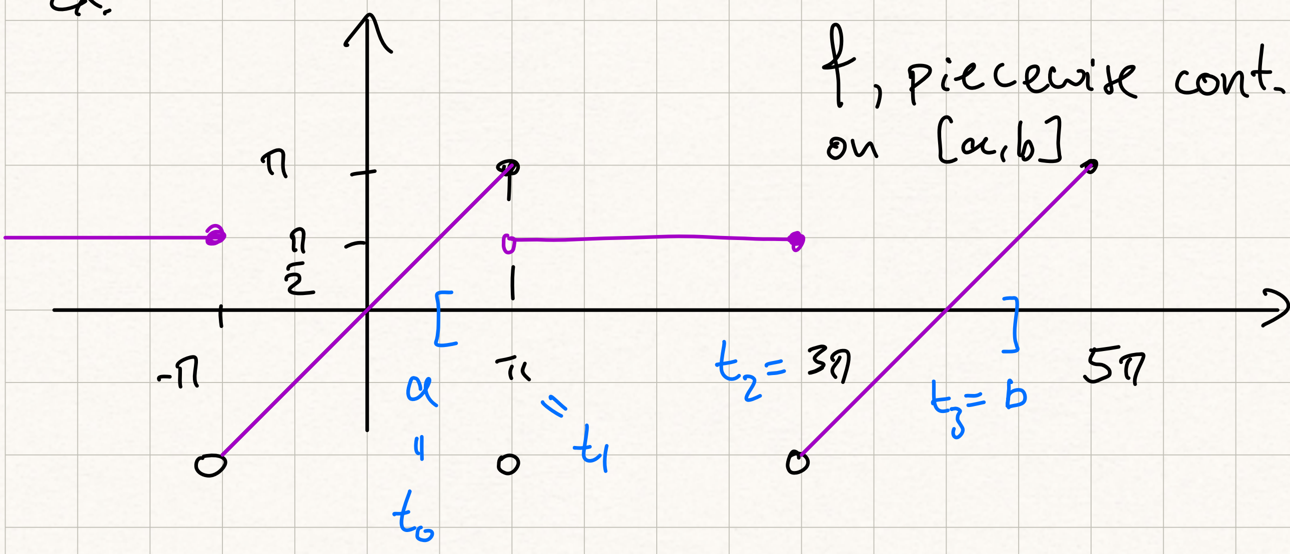
so side limit not finite, even though f cont. on $(0,1)$

Defn: A function f defined on an interval $[a,b]$ is called piecewise smooth if:

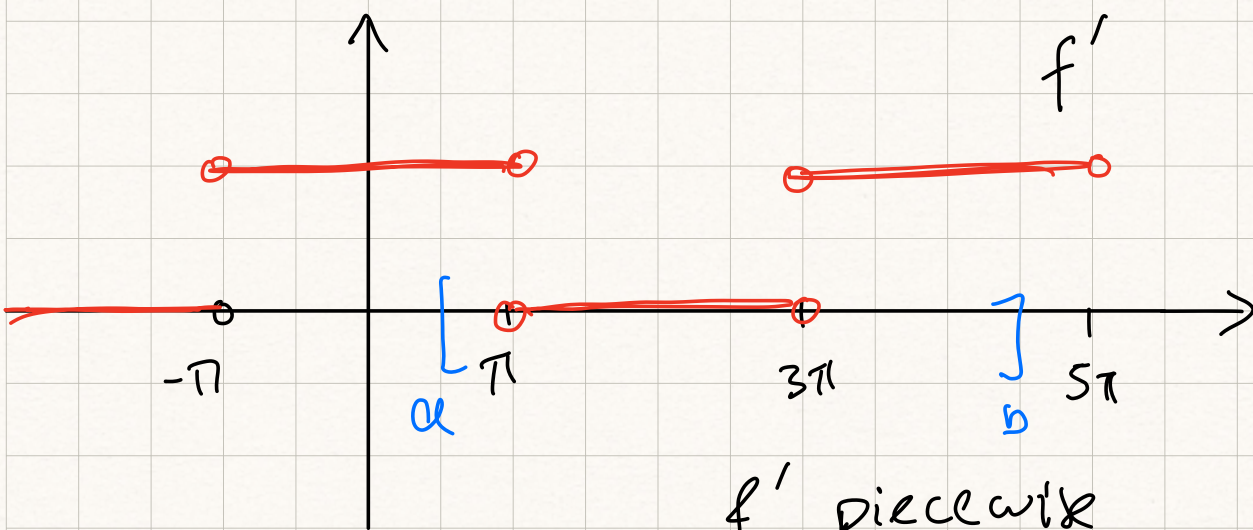
→ It is piecewise continuous on $[a,b]$

AND → Its derivative, defined away from points of discontinuity of f, is also piecewise continuous on $[a,b]$

Ex:



f, piecewise cont. on $[a,b]$



f'

f' piecewise cont.

<u>So:</u> f piecewise smooth

<u>Non-ex:</u> $f(t) = \sqrt{t}$, $t \in [0,1]$

cont. on $[0,1]$ $\Rightarrow$ piecewise continuous

<u>but</u> $f'(t) = \dfrac{1}{2\sqrt{t}}$ $t \in (0,1)$

which is <u>not</u> piecewise cont.

bec. $\lim\limits_{t \to 0^+} f(t) = \infty$

<u>Theorem:</u> If f periodic and on every interval $[a,b]$ it is piecewise smooth, then its F. S. converges:

a) to $f(t)$ if f is cont. at t.

b) to the average $\dfrac{f(t^+) + f(t^-)}{2}$ if f is discont. at t.

<u>Here:</u> $f(t^+) = \lim\limits_{s \to t^+} f(s)$

//