Lesson 28 03/25/2022

{9.2}

Theorem: If f periodic and on every interval $[a,b]$ it is piecewise smooth, then its F. S. converges:

[f piecewise cont. on $[a,b]$ and f' is piecewise cont. on $[a,b]$]

a) to $f(t)$ if f is cont. at t.

b) to the average $\dfrac{f(t^+)+f(t^-)}{2}$ if f is discont. at t.

Here: $f(t^+) = \lim_{s \to t^+} f(s)$ //

Observation: If f cont. at t then

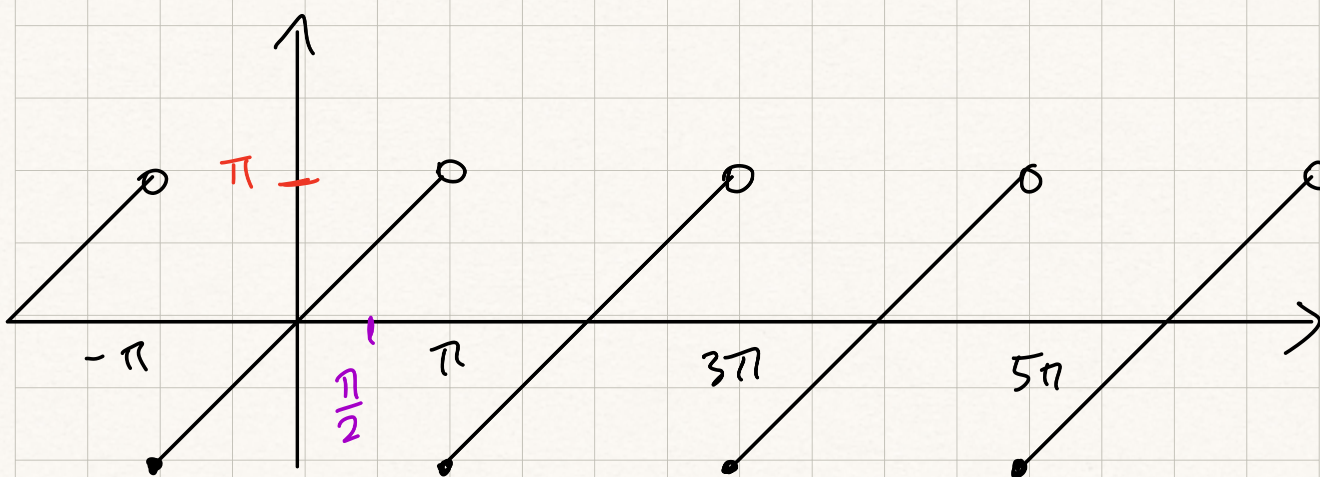
$$\lim_{s\to t^+} f(s) = \lim_{s \to t^-} f(s) = f(t)$$

So: $$\frac{f(t^+)+f(t^-)}{2} = \frac{f(t)+f(t)}{2} = f(t).$$

So: can say that F. S. converges to the average of side limits of f at t, regardless of whether f is cont. at t or not.

Ex: $f(t) = t$, $f \in \underbrace{[-\pi, \pi)}_{\text{interval of length } 2\pi}$

2π- periodic



$f \rightarrow$ piecewise smooth.

Found F.S. for f:

$$f \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nt)$$

For $t = 0$: f cont at 0. Expect F.S. to converge to $f(0) = 0$.

Plug in $t = 0$:

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(\cancelto{0}{n \cdot 0}) = 0 = f(0)$$

At $t=\pi$: f discont. here. Expect the F.S. to converge to

$$\frac{f(\pi^-)+f(\pi^+)}{2} = \frac{\pi+(-\pi)}{2} = 0$$

At $t=\frac{\pi}{2}$: f cont. at $\frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$.

Expect F.S. to converge to $\frac{\pi}{2}$

$$\Rightarrow \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right)$$

Can help us compute the value of an infinite sum!

n	$\sin\left(\frac{n\pi}{2}\right)$
$4k+1$	1
$4k+2$	0
$4k+3$	-1
$4k+4$	0

0 for even values of n

So:

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right)$$

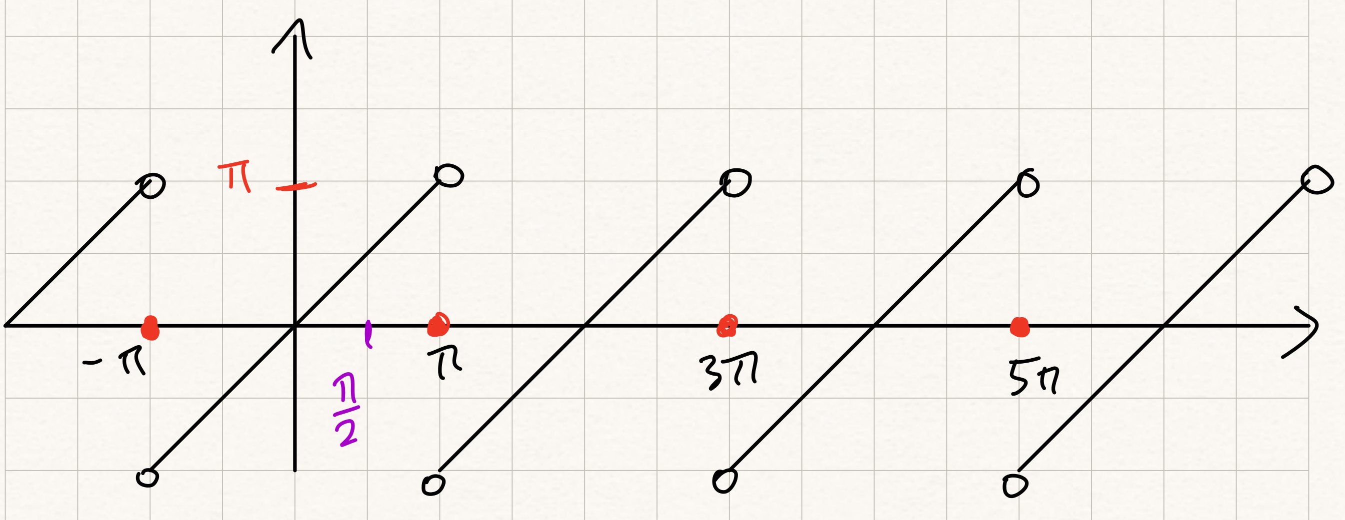
$$= \frac{2}{1}(-1)^{1+1}\cdot 1 + \frac{2}{2}(-1)^{2+1}\cdot 0 \to 0$$

$$+ \frac{2}{3}(-1)^{3+1}(-1)$$

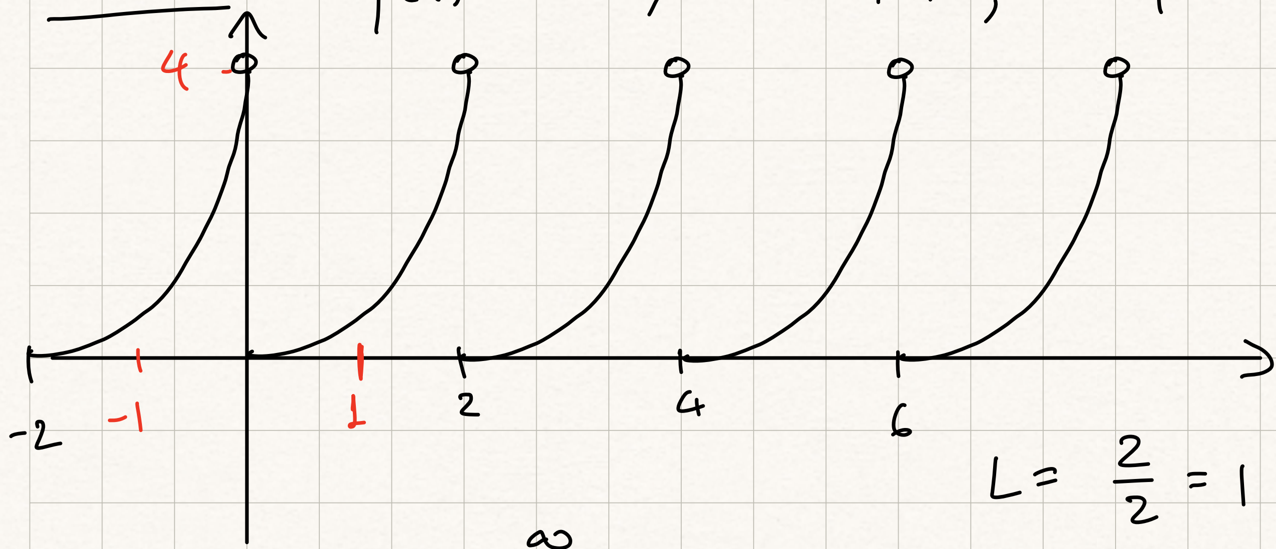
$$= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\right) = \frac{\pi}{2}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots = \frac{\pi}{4}.$$

Note: F.S. converges to a function which is different from the original one:



Ex 2: $f(t) = t^2$, $t \in [0,2)$, 2-periodic



$L = \frac{2}{2} = 1$

F.S.: $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\pi t) + b_n \sin(n\pi t)\right)$

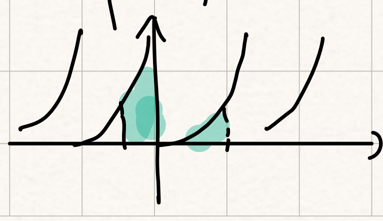
$$a_0 = \frac{1}{L}\int_0^{2L} f(t)\,dt = \int_0^2 t^2\,dt = \left.\frac{t^3}{3}\right|_0^2 = \frac{8}{3}$$

OR:

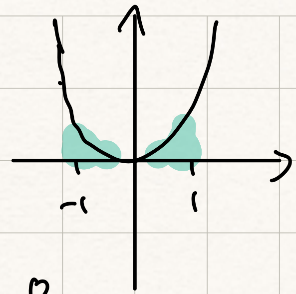
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \int_{-1}^{1} f(t)\,dt \neq \int_{-1}^{1} t^2\,dt$$

On $[0,1]$: $f(t) = t^2$

On $[-1,0)$: $f(t) = (t+2)^2$

$\int_{-1}^{1} f(t)\,dt \rightarrow$ 

$$\int_{-1}^{1} t^2\,dt \;\longrightarrow$$



<u>So:</u> $$a_0 = \int_{-1}^{0} (t+2)^2\,dt + \int_{0}^{1} t^2\,dt = \dots = \frac{8}{3}$$

$$a_n = \int_0^2 t^2 \cos(n\pi t)\,dt = \int_0^2 t^2 \frac{1}{\pi n}\left(\sin(n\pi t)\right)'\,dt$$

$$= \underbrace{\cancel{\frac{t^2 \sin(n\pi t)}{\pi n}\Bigg|_0^2}}_{0} - \int_0^2 \frac{2t}{\pi n} \sin(n\pi t)\,dt$$

$$= \int_0^2 \frac{2t}{(\pi n)^2}\left(\cos(n\pi t)\right)'\,dt$$

check: 0

$$= \frac{2t}{(\pi n)^2}\cos(n\pi t)\Bigg|_0^2 - \cancel{\int_0^2 \frac{2}{(\pi n)^2}\cos(n\pi t)\,dt}$$

$$= \frac{4\cos(n\pi\cdot 2)}{(\pi n)^2} = \frac{4}{(\pi n)^2}$$

<u>Similarly:</u> $$b_n = -\frac{4}{\pi n}$$

So

$$f \sim \frac{4}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{(\pi n)^2} \cos(n\pi t) - \frac{4}{n\pi} \sin(n\pi t) \right)$$

Plug in $t = 2$ into F.S.:

$$\frac{4+0}{2} = \frac{4}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{\pi^2 n^2} \underbrace{\cos(2\pi n)}_{=1} - \frac{4}{\pi n} \underbrace{\sin(2\pi n)}_{=0} \right)$$

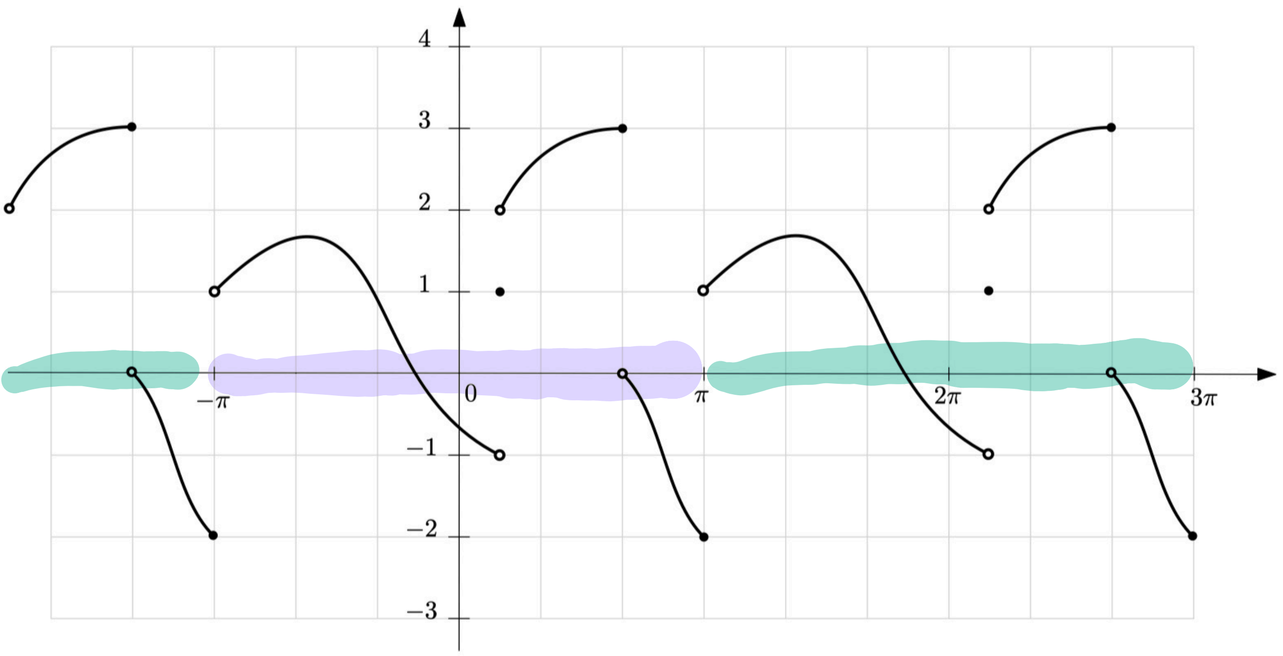
$$\Rightarrow \frac{2}{3} = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \Rightarrow$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

<u>Exercise:</u> Plug in $t=1$ to find $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Ex 3:

3. (4 pts.) You are given the graph of a 2π-periodic, piecewise smooth function $f(t)$.



Since it is 2π periodic and piecewise smooth, it has a Fourier series expansion of the form

$$f(t) \sim \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)}_{t = ?}.$$

Determine the value of the infinite sum

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n.$$

Note:

$$\sin(\pi n) = 0$$
$$\cos(\pi n) = (-1)^n$$

So:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n = \overbrace{\frac{f(\pi^-) + f(\pi^+)}{2}}^{\text{what F.S. converges to for } t = \pi}$$

$$= \frac{1 + (-2)}{2} = -\frac{1}{2}.$$