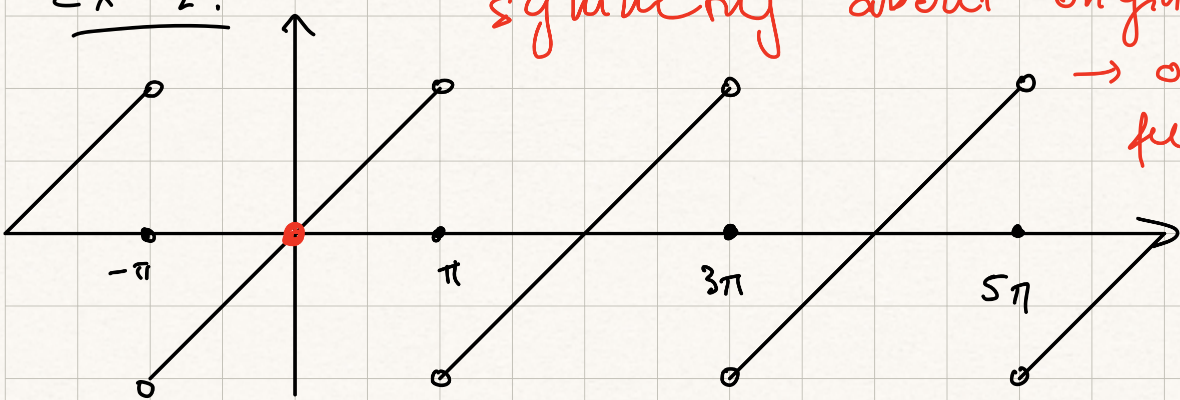
Lesson 29 03/28/22

9.3

Odd & Even functions

Ex 1:



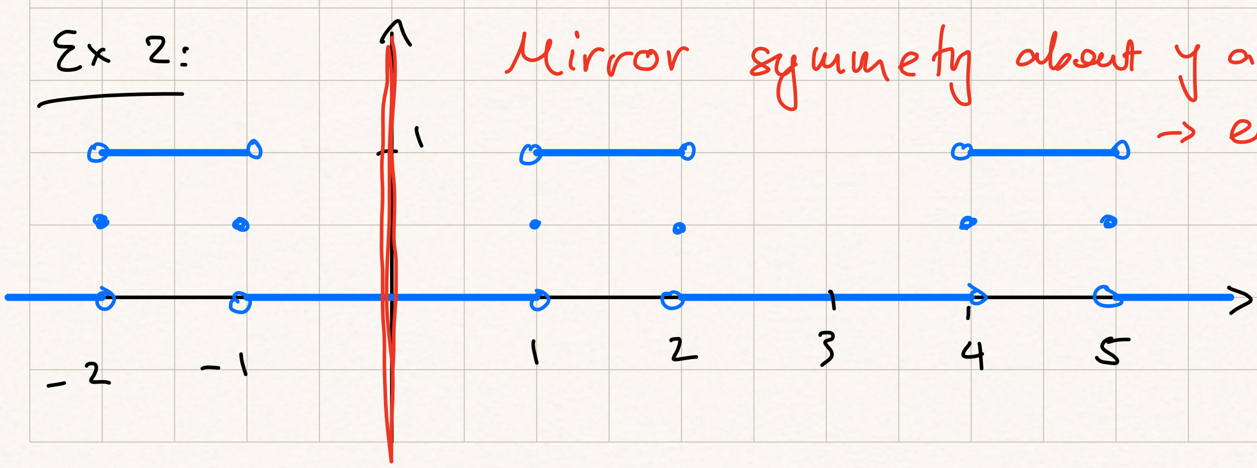
symmetry about origin → odd function

f piecewise smooth

Found: $f = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(n\pi t)$

Note: a_0, a_n, $n = 1, 2, 3 ..$ are all 0.

Ex 2:



Mirror symmetry about y axis → even

Found

$$f = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\left(\sin\left(\frac{4\pi n}{3}\right) - \sin\left(\frac{2\pi n}{3}\right)\right)\cos\left(\frac{2\pi n t}{3}\right)$$

Note: $b_n = 0$ for $n = 1, 2, \ldots$

Def'n: a) $f(t)$ is even if $f(t) = f(-t)$ for all t

Ex: $\cos(t)$ $\quad \cos(-t) = \cos(t)$

t^k, k even : $(-t)^k = t^k$

b) $f(t)$ is odd if $f(t) = -f(-t)$

Ex: $f(t) = \sin(t)$

$f(-t) = \sin(-t) = -\sin(t) = -f(t)$.

t^k, k odd

! A fct need not be odd or even:

$f(t) = 1 + t$

$f(-t) = 1 - t \neq 1 + t$ or $-(1+t)$

1. Even fcts have graphs symmetric wrt y axis.
2. Odd fcts have graphs symmetric about the origin.

Note: If f even:

$$\int_{-a}^{a} f(t)\,dt = \underbrace{\int_{-a}^{0} f(t)\,dt}_{s=-t} + \int_{0}^{a} f(t)\,dt$$

$$= -\int_{a}^{0} \underbrace{f(-s)}_{=f(s)}\,ds + \int_{0}^{a} f(t)\,dt$$

$$= \int_{0}^{a} f(s)\,ds + \int_{0}^{a} f(t)\,dt$$

$$= 2\int_{0}^{a} f(t)\,dt.$$

Exercise: If f is odd:

$$\int_{-a}^{a} f(t)\,dt = 0$$

How this relates to F.S.

Let $f(t)$ be p. smooth, $2L$-periodic, even

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} \underbrace{f(t)}_{\text{even}} \underbrace{\cos\left(\frac{n\pi}{L} t\right)}_{\text{even}} \, dt$$

$$= \frac{2}{L} \int_{0}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) dt$$

Check: even × even = even

$$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{f(t)}_{\text{even}} \underbrace{\sin\left(\frac{n\pi}{L} t\right)}_{\text{odd}} \, dt = 0$$

even × odd = odd

So: If f even & $2L$-periodic

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right)$$

$$a_n = \frac{2}{L} \int_{0}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) dt$$

$n = 0, 1, 2, \ldots$

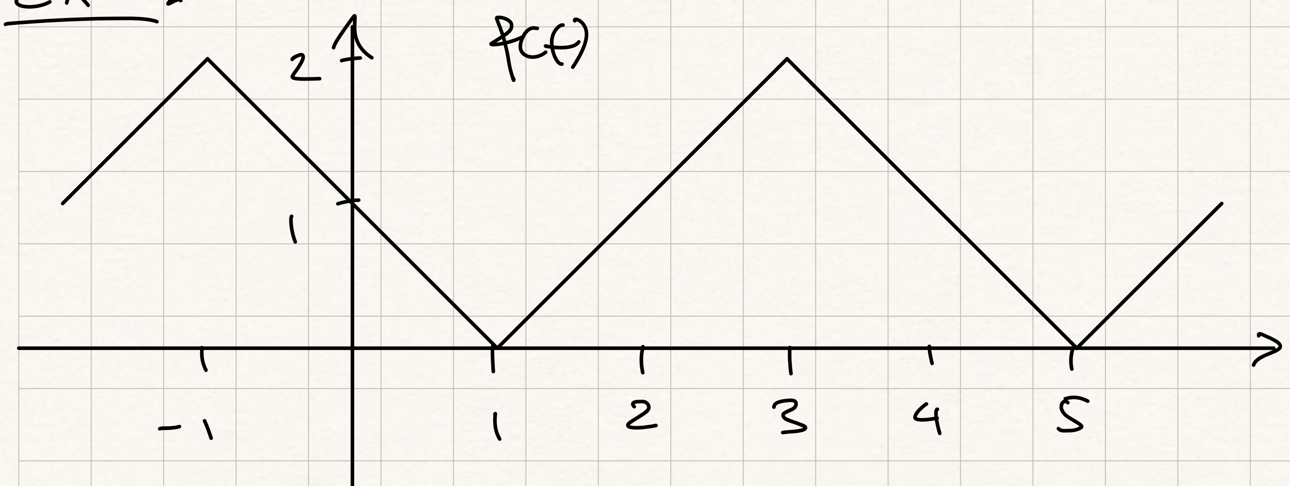
Similarly: if f odd & $2L$-periodic then F.S has only sine terms

odd × odd = even

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right)$$

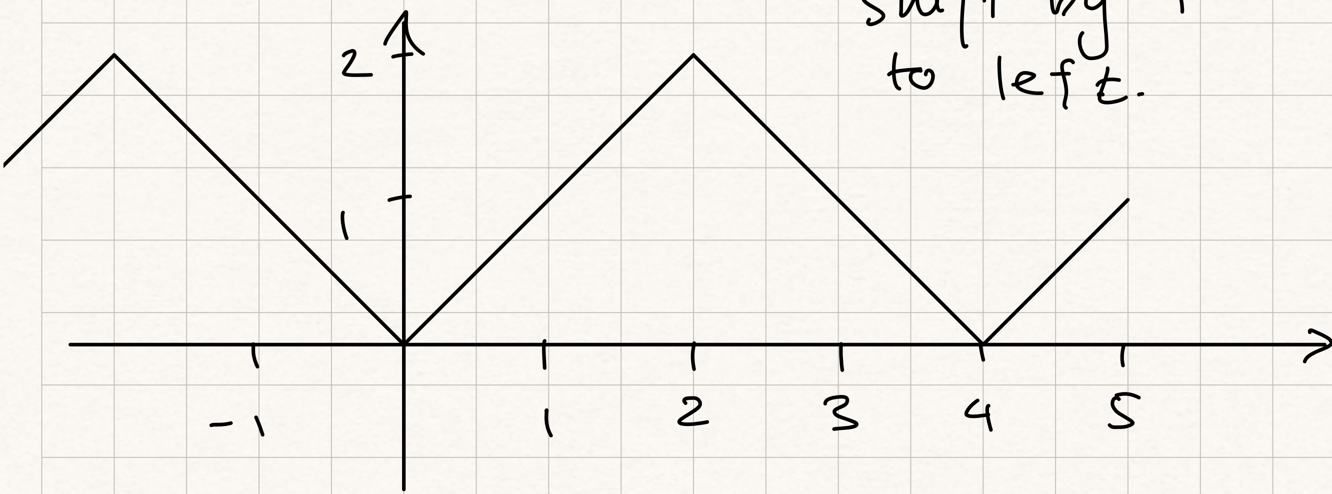
$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt$$

Ex 1



$f \to$ periodic w/ period 4.

Want: F.S.



shift by 1 to left.

$$g(t) = f(t+1)$$

$$L = 2$$

$$g = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2} t\right)$$

$$a_0 = \frac{2}{L}\int_0^L t\,dt = \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2$$

$$a_n = \frac{2}{L}\int_0^L t\cos\left(\frac{n\pi}{L}t\right) = \int_0^2 t\cos\left(\frac{n\pi}{2}t\right)dt$$

$$= \frac{2}{\pi n} t\sin\left(\frac{n\pi}{2}t\right)\Big|_0^2 - \frac{2}{\pi n}\int_0^2 \sin\left(\frac{n\pi}{2}t\right)dt$$

0

$$= \left(\frac{2}{\pi n}\right)^2 \cos\left(\frac{n\pi}{2}t\right)\Big|_0^2 = \left(\frac{2}{\pi n}\right)^2 \left(\cos(n\pi) - 1\right)$$

$$= \left(\frac{2}{\pi n}\right)^2 \left((-1)^n - 1\right)$$

So:

$$g(t) = \frac{2}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n}\right)^2 \left((-1)^n - 1\right) \cos\left(\frac{n\pi}{2}t\right)$$

$$f(t) = g(t-1)$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n}\right)^2 \left((-1)^n - 1\right) \cos\Big(\overbrace{\frac{n\pi}{2}t - \frac{n\pi}{2}}^{\frac{n\pi}{2}(t-1)}\Big)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n}\right)^2 \left((-1)^n - 1\right) \Big(\cos\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{2}t\right) + \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}t\right)\Big)$$

$$= 1 + \sum_{n=1}^{\infty} \Bigg[\overbrace{\left(\frac{2}{\pi n}\right)^2 \left((-1)^n - 1\right) \cos\left(\frac{n\pi}{2}\right)}^{a_n = 0} \cos\left(\frac{n\pi}{2}t\right) + \underbrace{\left(\frac{2}{\pi n}\right)^2 \left((-1)^n - 1\right) \sin\left(\frac{n\pi}{2}\right)}_{b_n} \sin\left(\frac{n\pi}{2}t\right) \Bigg]$$

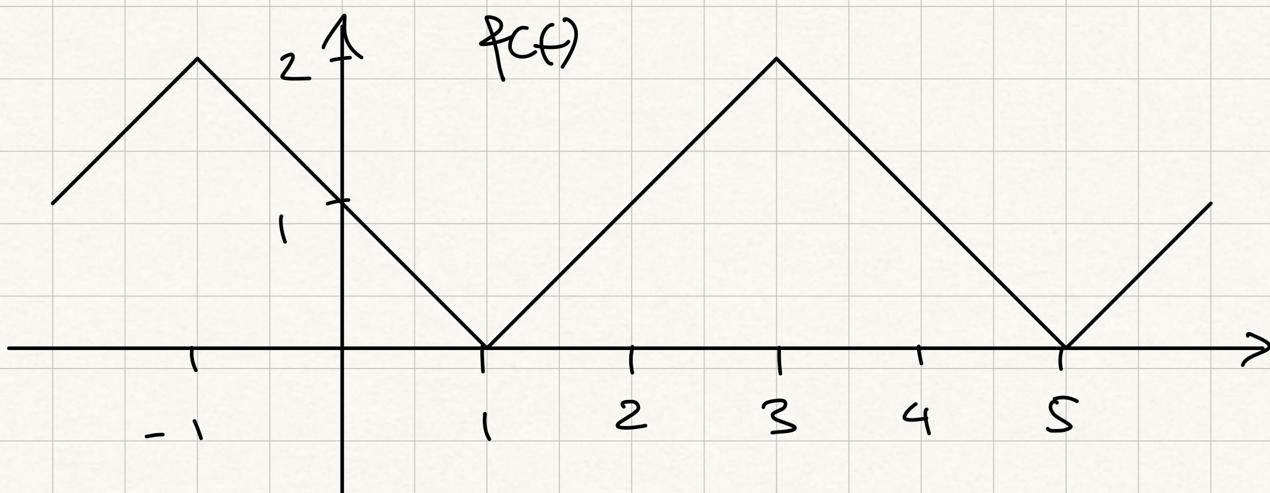
Note: If n odd: $\cos\left(\frac{n\pi}{2}\right) = 0$

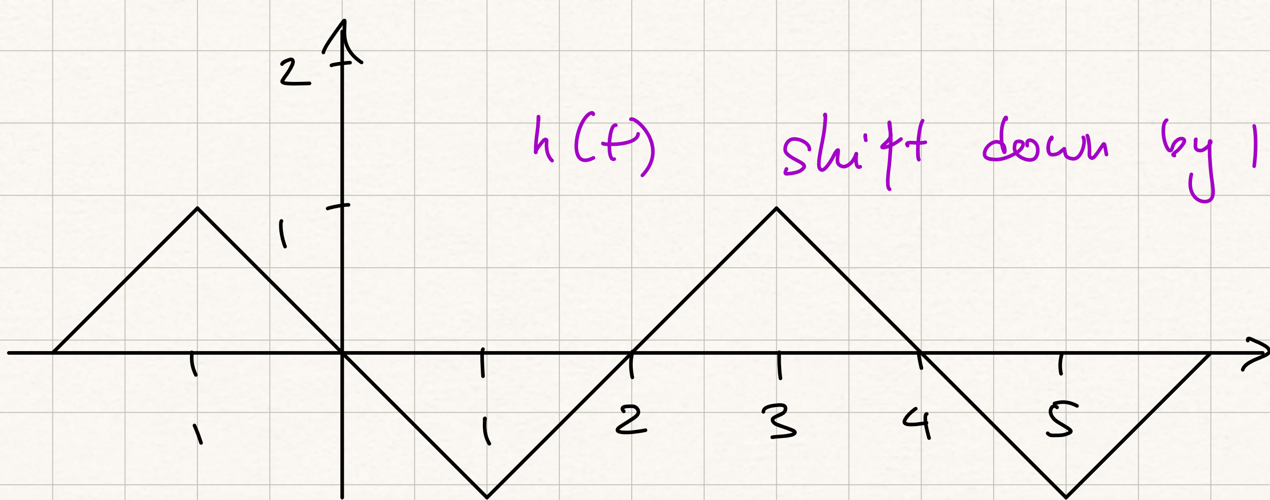
n even: $(-1)^n - 1 = 0$

So:

$$f(t) = 1 + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n}\right)^2 \left((-1)^n - 1\right) \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}t\right)$$

Another way to see this:





$h(t) = f(t) - 1$

$h \to$ odd!

$$h(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2} t\right)$$

$$\Rightarrow \quad f(t) = 1 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2} t\right)$$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi}{2} t\right) dt$$

$$= \int_0^2 f(t) \sin\left(\frac{n\pi}{2} t\right) dt$$

= break into 2

int'ls = ... $\boxtimes$