Lesson 3

01/14/2022

Last time:

- Structure of sol's
- Linear independence

Today: More on those, solve systems w/ real distinct eigenvalues.

From last time:

$$\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}} \quad \text{no nonhomog. term}$$

superposition: $\underline{\underline{x}}_1, \dots, \underline{\underline{x}}_n$ are sols $\rightarrow$

$c_1 \underline{\underline{x}}_1(t) + \dots + c_n \underline{\underline{x}}_n(t)$ is a sol'n.

Linear indep.: $\underline{\underline{x}}_1(t), \dots, \underline{\underline{x}}_n(t)$ lin. indep. on interval I if $c_1 \underline{\underline{x}}_1(t) + \dots + c_n \underline{\underline{x}}_n(t) = 0$ on I

$\Rightarrow c_1 = \dots = c_n = 0.$

Theorem: If $\underline{\underline{x}}_1(t), \dots, \underline{\underline{x}}_n(t)$ are linearly indep. sol's of the $n \times n$ system

$$\underline{\underline{x}}'(t) = \underline{\underline{P}}(t)\,\underline{\underline{x}}(t)$$

on an interval I, then <u>any</u> sol'n of the system is of the form

$$\underline{\underline{x}}(t) = c_1 \underline{\underline{x}}_1(t) + \dots + c_n \underline{\underline{x}}_n(t)$$

for some const. scalars $c_1, \dots, c_n$.

"lin. indep. sols are good building blocks"

How to check linear independence.

Sup. we have n vector valued fcts, each $n \times 1$ vector valued,

$$\underline{\underline{x}}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \ldots, \quad x_n(t) = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \\ \\ x_{nn}(t) \end{bmatrix}$$

Wronskian determinant:

$$W(\underline{\underline{x}}_1, \ldots, x_n)(t) = \det \begin{bmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & & x_{2n}(t) \\ \vdots & & \\ x_{n1}(t) & & x_{nn}(t) \end{bmatrix}$$

Criterion: If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are solutions of $\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}}$ on an interval I, then

→ If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are lin. independent on I then $W(\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n)(t) \neq 0$ for all $t \in I$.

→ If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are lin. dependent on I then $W(\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n)(t) = 0$ for all $t \in I$

Ex: $\underline{x}' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \underline{x}$

Given sols: $\underline{x}_1 = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{x}_2 = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Goal: find a sol'n on all of $\mathbb{R}$, w/
$\underline{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q: Is there such a sol'n? Is it unique?
Yes, by theorem on existence & uniqueness of sols.

Find sols:

Q: Are the sols $\underline{x}_1, \underline{x}_2$ lin. independent?

$$W(\underline{x}_1, \underline{x}_2)(t) = \det \begin{bmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{bmatrix}$$

$$= -2e^{5t} + e^{5t} = -e^{5t} \neq 0$$

for all $t \in \mathbb{R}$

$\Rightarrow$ x_1, x_2 lin. indep. on $\mathbb{R}$.

So any sol'n is of form
$$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t)$$

Want: $\underline{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\Rightarrow \quad c_1 \begin{bmatrix} e^{3\cdot 0} \\ -e^{3\cdot 0} \end{bmatrix} + c_2 \begin{bmatrix} e^{2\cdot 0} \\ -2e^{2\cdot 0} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ -c_1 - 2c_2 = 2 \end{cases} \Rightarrow \quad c_1 = 4, \; c_2 = -3$$

So:

$$\underline{\underline{x}}(t) = 4e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 3e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \quad //$$

Q: How do we find n lin. indep. sols?

Look at $\quad \underline{\underline{x}}'(t) = \underline{\underline{A}} \; \underline{\underline{x}}(t) \quad (*)$

$\underline{\underline{A}} \hookrightarrow$ const. matrix, $n \times n$

Ex: $\quad \underline{\underline{x}}' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \underline{\underline{x}}$

Remember: $\quad y'' - 5y' + 6y = 0$, set $y = e^{rt}$

plugged in

$$\rightarrow r^2 e^{rt} - 5re^{rt} + 6e^{rt} = 0$$

$$\Rightarrow (r^2 - 5r + 6)e^{rt} = 0$$

$$\Rightarrow r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$$

$$\Rightarrow y = e^{2t}, \; y = e^{3t} \text{ are sols.}$$

<u>Today:</u> guess that a sol'n to $*$ is of form $\underline{x}(t) = e^{\lambda t}\underline{v}$, for $\lambda, \underline{v}$ tbd

const. vector

$$\underline{x}(t) = e^{\lambda t}\underline{v} \Rightarrow \underline{x}'(t) = \lambda e^{\lambda t}\underline{v}$$

$$\underline{x}' = \underline{A}\,\underline{x} \Rightarrow \lambda e^{\lambda t}\underline{v} = \underline{A}\,e^{\lambda t}\underline{v}$$

$$\Rightarrow e^{\lambda t}(\underline{A}\,\underline{v} - \lambda\underline{v}) = \underline{0}$$

$$\Rightarrow \cancel{e^{\lambda t}}(\underline{A}\,\underline{v} - \lambda\underline{I}\,\underline{v}) = \underline{0}$$

identity matrix

$$\Rightarrow (\underline{A} - \lambda\underline{I})\,\underline{v} = \underline{0}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \ddots \end{bmatrix}$$

0 $n \times 1$ vector

So: <u>If</u> we can find λ such that there <u>is</u> vector $\underline{v} \neq \underline{0}$ w/ property

$$(\underline{A} - \lambda\underline{I})\,\underline{v} = \underline{0}$$

<u>Then</u>

$e^{\lambda t}\underline{v}$ will be a sol'n to $\underline{x}' = \underline{A}\,\underline{x}$.

Want: $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ to be non-invertible / have non-trivial nullspace / be singular.

so: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$ ← Charact. eqn of A.

$n \times n$ scalar $n \times n$ scalar

A λ (real, complex, 0) for which $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$ is called an **eigenvalue** of $\underline{\underline{A}}$.

An **eigenvector** associated to an eigenvalue λ is a non-zero vector $\underline{v}$ so that

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0} \iff \underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

Ex: $\underline{\underline{A}} = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$

Eigenvalues?

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det\left(\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix}\right) = (4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6 \Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow \lambda = 2, \ \lambda = 3$$

Eigenvalues: $\lambda = 2,\ \lambda = 3.$

Find: eigenvector(s) assoc. w/ $\lambda = 2$.

$$(\underline{\underline{A}} - 2\underline{\underline{I}})\,\underline{\underline{v}} = \underline{\underline{0}} \Rightarrow \begin{bmatrix} 4-2 & 1 \\ -2 & 1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2v_1 + v_2 = 0 \\ -2v_1 - v_2 = 0 \end{cases}$$

$$\Rightarrow v_2 = -2v_1$$

So: $\begin{bmatrix} v_1 \\ -2v_1 \end{bmatrix}$ is an e-vector for any $v_1 \neq 0$

Ex: $v_1 = 1 \Rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an e-vector,

$e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a sol'n to

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}.$$

Exercise: find e-vector for $\lambda = 3$.